A FINITE DIMENSIONAL L_∞ ALGEBRA EXAMPLE IN GAUGE THEORY

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Abstract. We construct an example of a finite dimensional L_∞ algebra which is generated by a Lie algebra together with a non-Lie action on another vector space. We then show how this example fits into the gauge transformation theory of Berends, Burgers and Van Dam.

Introduction

Although L_∞ algebras (or sh Lie algebras) have been objects of much research during the past several years, concrete examples of these structures remain somewhat elusive. Initial results by Daily [3] in the $\mathbb{Z}$ graded setting and by Fialowski and Penkava [5], [4], and by Bodin, Fialowski and Penkava [2] in the $\mathbb{Z}/2$ graded case have recently appeared. The original interest in these L_∞ structures was perhaps motivated by their appearance in several aspects of mathematical physics ranging from closed string field theory [10] to the gauge theory for fields on massless particles of high spin [1], [6].

In this note, we will recall the definition of L_∞ algebra structures in Section 1. In Section 2, we construct a finite dimensional example of such an algebra. We review the relationship between Berends, Burgers and van Dam's gauge theory and L_∞ algebras in Section 3. The mechanics of this correspondence is illustrated by using the example that is constructed in Section 2. This particular example consists of a Lie algebra together with its non-Lie action on another vector space. We leave it as a challenge to the physicists to develop a physical model whose gauge transformations are described by this algebraic example.

1. L_∞ algebras

We begin by recalling the definition of an L_∞ algebra [7],[9]. Let V be a graded vector space over a field k.

Definition 1. *An L_∞ structure on V is a collection of skew symmetric linear maps $l_n : V^{\otimes n} \to V$ of degree $2-n$ that satisfy the relations*

$$\sum_{i+j=n+1} \sum_{\sigma} e(\sigma)(-1)^{\sigma}(-1)^{i(j-1)} l_j(l_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}) = 0$$

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where $(-1)^\sigma$ is the sign of the permutation, $e(\sigma)$ is the sign that arises from the degrees of the permuted elements, and σ is taken over all $(i, n-i)$ unshuffles.

This is the cochain complex point of view; for chain complexes, we require that the maps l_n have degree $n-2$.

If we denote the desuspension of V by $\downarrow V$, i.e. $(\downarrow V)_n = V_{n+1}$, we may then describe an L_∞ structure on the cochain complex V by a coderivation $\overline{D}$ of degree $+1$ on the cocommutative coalgebra $\Lambda^*(\downarrow V)$ such that $\overline{D}^2 = 0$. Equivalently, this L_∞ structure may be described by the linear map $D : \Lambda^*(\downarrow V) \to \downarrow V$ where $D = p_1 \circ \overline{D}$ and $p_1 : \Lambda^*(\downarrow V) \to \downarrow V$ is the projection. From this point of view, the L_∞ algebra relations are given by $D \circ \overline{D} = 0$.

2. A finite dimensional example

Consider the graded vector space $V = \oplus V_n$ where V_0 is a 2 dimensional space with basis $< v_1, v_2 >$. and V_1 is a 1 dimensional space with basis $< w >$. Let $V_n = 0$ for $n \neq 0, 1$. We define an L_∞ structure, $l_n : V^{\otimes n} \to V$, on V via the following maps:

$$l_1(v_1) = l_1(v_2) = w$$
$$l_2(v_1 \otimes v_2) = v_1, \quad l_2(v_1 \otimes w) = w$$
$$l_n(v_2 \otimes w^{\otimes n-1}) = C_n w \quad \text{for all } n \geq 3$$

where $C_3 = 1$ and $C_n = (-1)^{n-1}(n-3)C_{n-1}$. We extend these maps to be skew symmetric and define l_n to be 0 when evaluated on any element of $V^{\otimes n}$ that is not listed above.

Theorem 2. *The maps l_n defined above satisfy the relations for an L_∞ algebra structure.*

Proof. It is clear that the first two relations, $l_1 l_1 = 0$ and $l_1 l_2 - l_2 l_1 = 0$ are satisfied. Moreover, the vector space V_0 with the bracket $[v_1, v_2] = l_2(v_1 \otimes v_2)$ is a Lie algebra. As a result, the next relation evaluated on $v_i \otimes v_j \otimes v_k$ is satisfied because the Jacobi identity holds here. However, the Jacobi identity does not hold when evaluated on the element $v_1 \otimes v_2 \otimes w$, but the generalized Jacobi expression $l_1 l_3 + l_2 l_2 + l_3 l_1$ will equal zero. This says that the action of V_0 on V_1 is not that of a Lie module. For $n > 3$, the summands in the L_∞ relation can be calculated as follows:

$$l_1 l_n(v_1 \otimes v_2 \otimes w^{\otimes n-2}) = 0$$

$$l_2 l_{n-1}(v_1 \otimes v_2 \otimes w^{\otimes n-2}) = (-1)^{n-1} l_2(l_{n-1}(v_2 \otimes w^{\otimes n-2}) \otimes v_1)$$

$$= (-1)^{n-1} C_{n-1} l_2(w \otimes v_1) = (-1)^n C_{n-1} w$$

$$l_k l_{n-k}(v_1 \otimes v_2 \otimes w^{\otimes n-2}) = 0 \text{ for all } 3 \leq k \leq n-3$$

because each summand in this expansion contains the term $l_k(v_1 \otimes w^{\otimes k-1}) = 0$. Further, we have

$$\begin{aligned} l_{n-1}l_2(v_1 \otimes v_2 \otimes w^{\otimes n-2}) &= l_{n-1}(l_2(v_1 \otimes v_2 \otimes w^{\otimes n-2}) - (n-2)l_{n-1}(l_2(v_1 \otimes w) \otimes v_2 \otimes w^{\otimes n-3}) \\ &= l_{n-1}(v_1 \otimes w^{\otimes n-2}) - (n-2)l_{n-1}(w \otimes v_2 \otimes w^{\otimes n-3}) \\ &= 0 + (n-2)l_{n-1}(v_2 \otimes w^{\otimes n-2}) = (n-2)C_{n-2}w. \\ l_n l_1(v_1 \otimes v_2 \otimes w^{\otimes n-2}) &= l_n(w \otimes v_2 \otimes w^{\otimes n-2}) - l_n(w \otimes v_1 \otimes w^{\otimes n-2}) = -C_n w. \end{aligned}$$

Consequently, the nth Jacobi expression is satisfied if and only if

$$\sum_{p=1}^{n}(-1)^{p(n-p)} l_{n-p+1} l_p((v_1 \otimes v_2 \otimes w^{\otimes n-2}) = 0$$

$$\Leftrightarrow (-1)^{(n-1)(1)}(-1)^n C_{n-1}w + (-1)^{2(n-2)}(n-2)C_{n-1}w + (-1)^{1(n-1)}(-1)C_n w = 0$$

$$\Leftrightarrow (-1)C_{n-1} + (n-2)C_{n-1} + (-1)^n C_n = 0$$

$$\Leftrightarrow C_n = (-1)^{n-1}(n-3)C_{n-1}.$$

□

3. The L_∞ Structure of a Gauge Algebra

Berends, Burgers, and van Dam [1] have described an algebraic framework for analyzing the action of field dependent gauge parameters on the space of fields for massless particles of high spin. In the classical setting, this action is usually that of the Lie algebra of vector fields acting on the Lie module of fields. However, the field dependence of the parameters in this case usually results in the loss of the strict Lie structure. This algebraic data was recast in [6] and was shown to lead to an L_∞ algebra structure on the space of parameters together with the space of fields. We may summarize this situation in the following manner. Let Ξ be the vector space of gauge parameters and Φ the vector space of fields. The "action" is given by a gauge transformation which may be interpreted as a linear map $\delta : \Xi \to Hom(\Lambda^*\Phi, \Phi)$ where $\Lambda^*\Phi$ is the cocommutative coalgebra generated by Φ. The map δ is then extended to a map $\hat{\delta} : Hom(\Lambda^*\Phi, \Xi) \to Hom(\Lambda^*\Phi, \Phi)$ via $\hat{\delta}(\pi) = ev \circ ((\delta \circ \pi) \otimes \mathbb{1}) \circ \Delta$ where ev is the evaluation map and Δ is the unshuffle comultiplication on $\Lambda^*\Phi$. Recall that the vector space $Hom(\Lambda^*\Phi, \Phi)$ has a canonical Lie bracket given by $[f, g] = f \circ \bar{g} - g \circ \bar{f}$ where $\bar{f}$ denotes the extension of a linear map $f \in Hom(\Lambda^*\Phi, \Phi)$ to a coderivation on $\Lambda^*\Phi$; recall that $\overline{f} = m \circ (f \otimes \mathbb{1}) \circ \Delta$ with m the product in the graded commutative algebra $\Lambda^*\Phi$.

Another ingredient in this scenario is the assumed existence of a map $\mathcal{C} : \Xi \otimes \Xi \to Hom(\Lambda^*\Phi, \Xi)$ that satisfies the (BBvD) hypothesis

$$[\delta(\xi), \delta(\eta)] = \hat{\delta}\mathcal{C}(\xi, \eta) \in Hom(\Lambda^*\Phi, \Phi)$$

for all $\xi, \eta \in \Xi$. After this map is extended to a map

$$\hat{\mathcal{C}} : Hom(\Lambda^*\Phi, \Xi) \otimes Hom(\Lambda^*\Phi, \Xi) \to Hom(\Lambda^*\Phi, \Xi)$$

via

$$\hat{\mathcal{C}}(\pi_1, \pi_2) = \mathcal{C} \circ ((\pi_1 \otimes \pi_2) \otimes \mathbb{1}) \circ (\Delta \otimes \mathbb{1}) \circ \Delta,$$

a bracket that satisfies the Jacobi identity may be imposed on the space $Hom\Lambda^*\Phi, \Xi)$ via

$$[\pi_1, \pi_2] = \pi_1 \circ \overline{\hat{\delta}(\pi_2)} - \pi_2 \circ \overline{\hat{\delta}(\pi_1)} + \hat{\mathcal{C}}(\pi_1, \pi_2).$$

Now consider the graded vector space V with $V_0 = \Xi, V_1 = \Phi$, and $V_n = 0$ for $n \neq 0, 1$. By Theorem 2 of [6], an L_∞ structure may be defined on V by constructing a linear map $D : \Lambda^*(\downarrow V) \to \downarrow V$ by piecing together the maps δ and $\mathcal{C}$. The (BBvD) hypothesis together with the Jacobi identity for the bracket on $Hom(\Lambda^*\Phi, \Xi)$ imply that $D \circ \bar{D} = 0$, the L_∞ relations for V.

We note that in [8], a different approach that uses symmetric brace algebras to exhibit the link between gauge transformations and L_∞ algebras is developed.

We now use the example presented in Section 2 to illustrate this gauge transformation point of view. Recall that the vector space under consideration is given by $V_0 =< v_1, v_2 >$ and $V_1 =< w >$. It is necessary to desuspend V so that the vectors v_0 and v_1 have degree -1 and w has degree 0. Define $\delta : V_0 \to Hom(\Lambda^*V_1, V_1)$ by

$$\delta(v_1)(w_1 \wedge \cdots \wedge w_n) = \begin{cases} -w, & \text{if } n = 0 \\ w, & \text{if } n = 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\delta(v_2)(w_1 \wedge \cdots \wedge w_n) = \begin{cases} -w & \text{if } n = 0 \\ 0 & \text{if } n = 1 \\ \overline{C}_{n+1} w & \text{if } n > 1 \end{cases}$$

where

$$\overline{C}_n = \begin{cases} (-1)^{\frac{n-2}{2}} C_n & \text{if } n \text{ is even} \\ (-1)^{\frac{n+1}{2}} C_n & \text{if } n \text{ is odd} \end{cases}$$

and each $w_i = w$. We also define $\mathcal{C} : V_0 \otimes V_0 \to Hom(\Lambda^*V_1, V_0)$ by $\mathcal{C}(v_1 \otimes v_2)(1) = v_1$, and by setting it equal to 0 otherwise. Extend $\mathcal{C}$ to all of $V_0 \otimes V_0$ by skew symmetry.

Theorem 3. *The maps δ and $\mathcal{C}$ satisfy the (BBvD) hypothesis; i.e.*

$$[\delta(v_1), \delta(v_2)] = \hat{\delta}\mathcal{C}(v_1, v_2).$$

Proof. In order to extend the image of the map δ to a coderivation on Λ^*V_1, we recall that the basis vector in V_1 has degree 0 in the desuspended complex. As a result, the unshuffle comultiplication on Λ^*V_1, has the form

$$\Delta(w_1 \wedge \cdots \wedge w_n) = \sum_{i=0}^{n} \binom{n}{i} (w_1 \wedge \cdots \wedge w_i) \otimes (w_{i+1} \wedge \cdots \wedge w_n).$$

As a result,

$$\overline{\delta(v_1)}(w_1 \wedge \cdots \wedge w_n) = -w \wedge w_1 \wedge \cdots \wedge w_n + n(w_1 \wedge \cdots \wedge w_n)$$

and

$$\overline{\delta(v_2)}(w_1 \wedge \cdots \wedge w_n) = \begin{cases} -w \wedge w_1 \wedge \cdots \wedge w_n & \text{for } i = 0 \\ 0 & \text{for } i = 1 \\ \sum_{i=2}^n \binom{n}{i} \overline{C}_{i+1} w \wedge w_{i+1} \wedge \cdots \wedge w_n & \text{for } i > 1. \end{cases}$$

Consequently, we have

$$\delta(v_1) \circ \overline{\delta(v_2)}(w_1 \wedge \cdots \wedge w_n) = \begin{cases} -w & \text{if } n = 0 \\ 0 & \text{if } n = 1 \\ \overline{C}_{n+1} w & \text{if } n > 1 \end{cases}$$

and

$$\delta(v_2) \circ \overline{\delta(v_1)}(w_1 \wedge \cdots \wedge w_n) = \begin{cases} 0 & \text{if } n = 0 \\ -w & \text{if } n = 1 \\ (-\overline{C}_{n+2} + n\overline{C}_{n+1})w & \text{if } n > 1. \end{cases}$$

When $n > 1$, $[\delta(v_1), \delta(v_2)](w_1 \wedge \cdots \wedge w_n)$

$$= (\delta(v_1) \circ \overline{\delta(v_2)} - \delta(v_2) \circ \overline{\delta(v_1)})(w_1 \wedge \cdots \wedge w_n)$$

$$= \overline{C}_{n+1} w - ((-\overline{C}_{n+2} + n\overline{C}_{n+1})w).$$

But,

$$\overline{C}_{n+1} + -\overline{C}_{n+2} - n\overline{C}_{n+1}$$

$$= \begin{cases} (1-n)(-1)^{\frac{n+2}{2}} C_{n+1} + (-1)^{\frac{n}{2}} C_{n+2} & \text{if } n \text{ is even} \\ (1-n)(-1)^{\frac{n-1}{2}} C_{n+1} + (-1)^{\frac{n+3}{2}} C_{n+2} & \text{if } n \text{ is odd} \end{cases}$$

$$= \begin{cases} (1-n)(-1)^{\frac{n+2}{2}} C_{n+1} + (-1)^{\frac{n}{2}} (-1)^{n+1}(n-1) C_{n+1} & \text{if } n \text{ is even} \\ (1-n)(-1)^{\frac{n-1}{2}} C_{n+1} + (-1)^{\frac{n+3}{2}} (-1)^{n+1}(n-1) C_{n+1} & \text{if } n \text{ is odd.} \end{cases}$$

We note that $(-1)^{\frac{n+2}{2}} - (-1)^{\frac{n}{2}}(-1)^{n+1} = 0$ when n is even, and that $(-1)^{\frac{n-1}{2}} - (-1)^{\frac{n+3}{2}}(-1)^{n+1} = 0$ when n is odd.

In summary, we obtain

$$[\delta(v_1), \delta(v_2)](w_1 \wedge \cdots \wedge w_n) = \begin{cases} -w & \text{when } n = 0 \\ w & \text{when } n = 1 \\ 0 & \text{when } n > 1. \end{cases} \tag{1}$$

On the other hand,

$$\hat{\delta}\mathcal{C}(v_1, v_2)(w_1 \wedge \cdots \wedge w_n) = ev \circ (\delta \circ \mathcal{C}(v_1, v_2) \otimes \mathbb{1})(\sum_{i=0}^n \binom{n}{i} (w_1 \wedge \cdots \wedge w_i) \otimes (w_{i+1} \wedge \cdots \wedge w_n))$$

$$= ev \circ (\delta(v_1) \otimes \mathbb{1})(\sum_{i=0}^n \binom{n}{i} (w_1 \wedge \cdots \wedge w_i) \otimes (w_{i+1} \wedge \cdots \wedge w_n))$$

$$= \delta(v_1)(w_1 \wedge \cdots \wedge w_n) = \begin{cases} -w & \text{when } n = 0 \\ w & \text{when } n = 1 \\ 0 & \text{when } n > 1 \end{cases}$$

which agrees with equation 1.

□

Remark 4. *The fact that the coefficients C_n and $\overline{C}_n$ differ by only a sign that depends on n is due to the difference in the gradings of $V^{\otimes n}$ and $(\downarrow V)^{\otimes n}$. Details regarding this sign adjustment may be found in* [7] *and* [9]*.*

4. CODA

As was shown in Section 2, it is possible for a "small" graded vector space to carry a rich L_∞ algebra structure. Other similar examples have been constructed and will be discussed elsewhere. These examples in the 2-graded context include include the cases in which V_0 is an abelian Lie algebra but V_1 is not a V_0 module, and V_0 is a Lie algebra and V_1 is a V_0 module yet non trivial higher order operations l_n exist. These examples are interesting in their own right, but as seen in Section 3, the higher order operations can be used to construct the gauge transformations in the Berends, Burgers,van Dam theory. It is our hope that these algebra results may lead to new insights in the physical theories.

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