A FINITE DIMENSIONAL A_∞ ALGEBRA EXAMPLE

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Dedicated to Tornike Kadeishvili on the occasion of his 60th birthday

Abstract. We construct an example of an A_∞ algebra structure defined over a finite dimensional graded vector space.

Introduction

A_∞ algebras (or sha algebras) and L_∞ (or sh Lie algebras) have been topics of current research. Construction of small examples of these algebras can play a role in gaining insight into deeper properties of these structures. These examples may prove useful in developing a deformation theory as well as a representation theory for these algebras.

In [2], an L_∞ algebra structure on the graded vector space $V = V_0 \oplus V_1$ where V_0 is a 2 dimensional vector space, and V_1 is a 1 dimensional space, is discussed. This surprisingly rich structure on this small graded vector space was shown by Kadeishvili and Lada, [3], to be an example of an open-closed homotopy algebra (OCHA) defined by Kajiura and Stasheff [4]. In an unpublished note [1] M. Daily constructs a variety of other L_∞ algebra structures on this same vector space.

In this article we add to this collection of structures on the vector space V by providing a detailed construction of non-trivial A_∞ algebra data for V.

1. A_∞ Algebras

We first recall the definition of an A_∞ algebra (Stasheff [6]).

Definition 1.1. *Let V be a graded vector space. An A_∞ structure on V is a collection of linear maps $m_k : V^{\otimes k} \to V$ of degree $2-k$ that satisfy the identity*

$$\sum_{\lambda=0}^{n-1}\sum_{k=1}^{n-\lambda} \alpha m_{n-k+1}(x_1 \otimes \cdots \otimes x_\lambda \otimes m_k(x_{\lambda+1} \otimes \cdots \otimes x_{\lambda+k}) \otimes x_{\lambda+k+1} \otimes \cdots \otimes x_n) = 0$$

where $\alpha = (-1)^{k+\lambda+k\lambda+kn+k(|x_1|+\cdots+|x_\lambda|)}$, for all $n \geq 1$.

This utilizes the cochain complex convention. One may alternatively utilize the chain complex convention by requiring each map m_k to have degree $k-2$.

We will define the desuspension of V (denoted $\downarrow V$) as the graded vector space with indices given by $(\downarrow V)_n = V_{n+1}$, and the desuspension operator, $\downarrow : V \to \downarrow V$ (resp. suspension operator $\uparrow : \downarrow V \to V$) in the natural sense.

Stasheff also showed that an A_∞ structure on V is equivalent to the existence of a degree 1 coderivation $D : T^* \downarrow V \to T^* \downarrow V$ with the property $D^2 = 0$. Here, $T^* \downarrow V$ is the tensor coalgebra on the graded vector space $\downarrow V$.

Such a coderivation is constructed by defining

$D := \sum_{k=1}^{\infty} m'_k$, where $m'_k : \downarrow V^{\otimes k} \to \downarrow V$ is given by first defining $m'_k := (-1)^{\frac{k(k-1)}{2}} \downarrow \circ m_k \circ \uparrow^{\otimes k}$ and then extending each m'_k to a coderivation on $T^* \downarrow V$.

2. A Finite Dimensional Example

Let V denote the graded vector space given by $V = \bigoplus V_n$ where V_0 has basis $< v_1, v_2 >$, V_1 has basis $< w >$, and $V_n = 0$ for $n \neq 0, 1$. Define a structure on V by the following linear maps $m_n : V^{\otimes n} \to V$:

$$m_1(v_1) = m_1(v_2) = w$$
$$\textit{For } n \geq 2: \quad m_n(v_1 \otimes w^{\otimes k} \otimes v_1 \otimes w^{\otimes (n-2)-k}) = (-1)^k s_n v_1, \ 0 \leq k \leq n-2$$
$$m_n(v_1 \otimes w^{\otimes (n-2)} \otimes v_2) = s_{n+1} v_1$$
$$m_n(v_1 \otimes w^{\otimes (n-1)}) = s_{n+1} w$$

where $s_n = (-1)^{\frac{(n+1)(n+2)}{2}}$, and $m_n = 0$ when evaluated on any element of $V^{\otimes n}$ that is not listed above.

Theorem 2.1. *The maps defined above give the graded vector space V an A_∞ algebra structure.*

It is worth noting that this assumes the cochain convention regarding A_∞ algebra structures. The proof of this theorem relies on two lemmas:

Lemma 2.2. *Let $m'_n : \downarrow V^{\otimes n} \to \downarrow V := (-1)^{\frac{n(n-1)}{2}} \downarrow \circ m_n \circ \uparrow^{\otimes n}$ where $\downarrow V$ denotes the desuspension of V. Under the preceding definitions for m_n and V, we have the following definitions for m'_n:*

$$m'_1 = \downarrow m_1$$
$$\textit{For } n \geq 2: \quad m'_n(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-2)-k}) = \downarrow v_1, \ 0 \leq k \leq n-2$$
$$m'_n(\downarrow v_1 \otimes \downarrow w^{\otimes (n-2)} \otimes \downarrow v_2) = \downarrow v_1$$
$$m'_n(\downarrow v_1 \otimes \downarrow w^{\otimes (n-1)}) = \downarrow w$$

Remark 2.3. *Each m'_n is of degree 1.*

Lemma 2.4. *Let $D = \sum_{k=1}^{\infty} m'_k$ where m'_k is defined above. Let $n \geq 2$ be a positive integer. Suppose $D^2(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_m) = 0 \ \forall \ x_i \in V,\ 1 \leq m \leq n-1$.*

Then $$D^2(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) = \sum_{i+j=n+1} m'_i m'_j(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n)$$

Proof of Lemma 2.2. $m'_1(x) = (-1)^0 \downarrow \circ m_1 \circ \uparrow (\downarrow x) = \downarrow m_1(x)$ for any x.

Now let $n \geq 2$. The majority of the work here is centered around computing the signs associated with the graded setting. The elements x_i and the maps $\uparrow$, $\downarrow$, and m_n all contribute to an overall sign via their degrees. Observing these signs, we find

$$\begin{aligned} m'_n(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) &= (-1)^{\frac{n(n-1)}{2}} \downarrow \circ m_n \circ \uparrow^{\otimes n} (\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) \\ &= \begin{cases} (-1)^{\sum_{i=1}^{n/2} |x_{2i-1}|} \downarrow m_n(x_1 \otimes x_2 \otimes \cdots \otimes x_n) & \text{if n is even.} \\ (-1)^{\sum_{i=1}^{(n-1)/2} |x_{2i}|} \downarrow m_n(x_1 \otimes x_2 \otimes \cdots \otimes x_n) & \text{if n is odd.} \end{cases} \end{aligned}$$

First consider $m'_n(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-2)-k}),\ 0 \leq k \leq n-2$:

Case 1: n is even, k is even. Then

$$\begin{aligned} m'_n(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-2)-k}) &= (-1)^{|v_1| + (\frac{n}{2}-1)|w|} \downarrow m_n(v_1 \otimes w^{\otimes k} \otimes v_1 \otimes w^{\otimes (n-2)-k}) \\ &= (-1)^{0+\frac{n}{2}-1}(-1)^k s_n \downarrow v_1 \\ &= (-1)^{\frac{n}{2}-1}(-1)^{\frac{(n+1)(n+2)}{2}} \downarrow v_1 \\ &= (-1)^{\frac{n}{2}-1}(-1)^{(n+1)(\frac{n}{2}+1)} \downarrow v_1 \quad (*) \end{aligned}$$

If $\frac{n}{2}$ is even, then $(*) = (-1)^{\text{odd}}(-1)^{\text{odd*odd}} \downarrow v_1 = \downarrow v_1$.
If $\frac{n}{2}$ is odd, then $(*) = (-1)^{\text{even}}(-1)^{\text{odd*even}} \downarrow v_1 = \downarrow v_1$.

Case 2: n is even, k is odd. Then

$$\begin{aligned} m'_n(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-2)-k}) &= (-1)^{2|v_1| + (\frac{n}{2}-2)|w|} \downarrow m_n(v_1 \otimes w^{\otimes k} \otimes v_1 \otimes w^{\otimes (n-2)-k}) \\ &= (-1)^{0+\frac{n}{2}-2}(-1)^k s_n \downarrow v_1 \\ &= -(-1)^{\frac{n}{2}}(-1)^{\frac{(n+1)(n+2)}{2}} \downarrow v_1 \\ &= -(-1)^{\frac{n}{2}}(-1)^{(n+1)(\frac{n}{2}+1)} \downarrow v_1 \quad (**) \end{aligned}$$

If $\frac{n}{2}$ is even, then $(**) = -(-1)^{\text{even}}(-1)^{\text{odd*odd}} \downarrow v_1 = \downarrow v_1$.
If $\frac{n}{2}$ is odd, then $(**) = -(-1)^{\text{odd}}(-1)^{\text{odd*even}} \downarrow v_1 = \downarrow v_1$.

Case 3: n is odd, k is even. Then

$$\begin{aligned} m_n'(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-2)-k}) &= (-1)^{|v_1| + (\frac{n-1}{2} - 1)|w|} \downarrow m_n(v_1 \otimes w^{\otimes k} \otimes v_1 \otimes w^{\otimes (n-2)-k}) \\ &= (-1)^{0 + \frac{n-1}{2} - 1} (-1)^k s_n \downarrow v_1 \\ &= (-1)^{\frac{n-1}{2} - 1} (-1)^{\frac{(n+1)(n+2)}{2}} \downarrow v_1 \\ &= -(-1)^{\frac{n-1}{2}} (-1)^{\frac{(n+1)}{2}(n+2)} \downarrow v_1 \quad (***) \end{aligned}$$

If $\frac{n-1}{2}$ is even, then $(***) = -(-1)^{\text{even}}(-1)^{\text{odd*odd}} \downarrow v_1 = \downarrow v_1$.
If $\frac{n-1}{2}$ is odd, then $(***) = -(-1)^{\text{odd}}(-1)^{\text{even*odd}} \downarrow v_1 = \downarrow v_1$.

Case 4: n is odd, k is odd. Then

$$\begin{aligned} m_n'(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-2)-k}) &= (-1)^{(\frac{n-1}{2})|w|} \downarrow m_n(v_1 \otimes w^{\otimes k} \otimes v_1 \otimes w^{\otimes (n-2)-k}) \\ &= (-1)^{\frac{n-1}{2}} (-1)^k s_n \downarrow v_1 \\ &= -(-1)^{\frac{n-1}{2}} (-1)^{\frac{(n+1)(n+2)}{2}} \downarrow v_1 \\ &= -(-1)^{\frac{n-1}{2}} (-1)^{\frac{(n+1)}{2}(n+2)} \downarrow v_1 \quad (****) \end{aligned}$$

If $\frac{n-1}{2}$ is even, then $(****) = -(-1)^{\text{even}}(-1)^{\text{odd*odd}} \downarrow v_1 = \downarrow v_1$.
If $\frac{n-1}{2}$ is odd, then $(****) = -(-1)^{\text{odd}}(-1)^{\text{even*odd}} \downarrow v_1 = \downarrow v_1$.

Hence $m_n'(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-2)-k}) = \downarrow v_1, \ 0 \le k \le n-2$

Now consider $m_n'(\downarrow v_1 \otimes \downarrow w^{\otimes (n-2)} \otimes \downarrow v_2)$:

Case 1: n is even. Then

$$\begin{aligned} m_n'(\downarrow v_1 \otimes \downarrow w^{\otimes (n-2)} \otimes \downarrow v_2) &= (-1)^{|v_1| + (\frac{n}{2} - 1)|w|} m_n(\downarrow v_1 \otimes \downarrow w^{\otimes (n-2)} \otimes \downarrow v_2) \\ &= (-1)^{\frac{n}{2} - 1} s_{n+1} \downarrow v_1 \\ &= (-1)^{\frac{n}{2} - 1} (-1)^{\frac{(n+2)(n+3)}{2}} \downarrow v_1 \\ &= (-1)^{\frac{n}{2} - 1} (-1)^{(\frac{n}{2} - 1)(n+3)} \downarrow v_1 \quad (*) \end{aligned}$$

If $\frac{n}{2}$ is even, then $(*) = (-1)^{\text{odd}}(-1)^{\text{odd*odd}} \downarrow v_1 = \downarrow v_1$.
If $\frac{n}{2}$ is odd, then $(*) = (-1)^{\text{even}}(-1)^{\text{even*odd}} \downarrow v_1 = \downarrow v_1$.

Case 2: n is odd. Then

$$\begin{aligned} m'_n(\downarrow v_1 \otimes \downarrow w^{\otimes(n-2)} \otimes \downarrow v_2) &= (-1)^{(\frac{n-1}{2})|w|} m_n(\downarrow v_1 \otimes \downarrow w^{\otimes(n-2)} \otimes \downarrow v_2) \\ &= (-1)^{\frac{n-1}{2}} s_{n+1} \downarrow v_1 \\ &= (-1)^{\frac{n-1}{2}} (-1)^{\frac{(n+2)(n+3)}{2}} \downarrow v_1 \\ &= (-1)^{\frac{n-1}{2}} (-1)^{(n+2)(\frac{n+3}{2})} \downarrow v_1 \quad (**) \end{aligned}$$

If $\frac{n-1}{2}$ is even, then $(**) = (-1)^{\text{even}}(-1)^{\text{odd*even}} \downarrow v_1 = \downarrow v_1$.
If $\frac{n-1}{2}$ is odd, then $(**) = (-1)^{\text{odd}}(-1)^{\text{odd*odd}} \downarrow v_1 = \downarrow v_1$.

Hence $m'_n(\downarrow v_1 \otimes \downarrow w^{\otimes(n-2)} \otimes \downarrow v_2) = \downarrow v_1$

The preceding arguments for cases 1 and 2 for $m'_n(\downarrow v_1 \otimes \downarrow w^{\otimes(n-2)} \otimes \downarrow v_2)$ may be repeated for $m'_n(\downarrow v_1 \otimes \downarrow w^{\otimes(n-1)})$.

Thus $m'_n(\downarrow v_1 \otimes \downarrow w^{\otimes(n-1)}) = \downarrow w$ □

Proof of Lemma 2.4. We first note that

$$D^2(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) = \sum_{i+j \le n+1} m'_i m'_j(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n)$$

since $m'_k(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_l) = 0$ for $k > l$. So

$$\begin{aligned} D^2(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) &= \sum_{i+j \le n} m'_i m'_j(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) \\ &\quad + \sum_{i+j = n+1} m'_i m'_j(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) \end{aligned}$$

Hence it suffices to show that $\sum_{i+j \le n} m'_i m'_j(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) = 0$

Consider $\sum_{i+j \le n} m'_i m'_j(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n)$: Since $i + j \le n$, we can break this sum up into 4 different types of of elements in $\downarrow V^{\otimes k}$ based on whether the first and last terms in the tensor product contain m'_i or m'_j:

- Type 1: Elements with first term $\downarrow x_1$ and last term $\downarrow x_n$
  (example: $\downarrow x_1 \otimes \downarrow x_2 \otimes m'_1(\downarrow x_3) \otimes m'_2(\downarrow x_4 \otimes \downarrow x_5) \otimes \downarrow x_6$)
- Type 2: Elements with first term $\downarrow x_1$ and last term containing m'_k for some k
  (example: $\downarrow x_1 \otimes \downarrow x_2 \otimes m'_3(\downarrow x_3 \otimes m'_2(\downarrow x_4 \otimes \downarrow x_5) \otimes \downarrow x_6)$)
- Type 3: Elements with first term containing m'_k for some k and last term $\downarrow x_n$

(example: $m'_2(\downarrow x_1 \otimes \downarrow x_2) \otimes m'_1(\downarrow x_3) \otimes \downarrow x_4 \otimes \downarrow x_5 \otimes \downarrow x_6)$

- Type 4: Elements with first term containing m'_k and last term containing m'_l for some k, l
  (example: $m'_2(\downarrow x_1 \otimes \downarrow x_2) \otimes \downarrow x_3 \otimes \downarrow x_4 \otimes m'_2(\downarrow x_5 \otimes \downarrow x_6))$

Now each term of type 1 must be produced by $m'_i m'_j$ with $i + j \leq n - 1$. Hence, by factorization of tensor products, all possible terms of type 1 are given by:

$$\begin{aligned}(-1)^{2|x_1|-2}\Big(\downarrow x_1 \otimes \big(\sum_{i+j\leq n-1} m'_i m'_j(\downarrow x_2 \otimes \downarrow x_3 \otimes \cdots \otimes \downarrow x_{n-1})\big)\Big) \otimes \downarrow x_n \\ = \Big(\downarrow x_1 \otimes \big(D^2(\downarrow x_2 \otimes \downarrow x_3 \otimes \cdots \otimes \downarrow x_{n-1})\big)\Big) \otimes \downarrow x_n \\ = \big(\downarrow x_1 \otimes 0\big) \otimes \downarrow x_n \\ = 0\end{aligned}$$

since $D^2 = 0$ when evaluated on $n - 2$ terms.

Now since all terms of type 1 form a collection of elements in $\downarrow V^{\otimes k}$ that sum up to 0, we can duplicate this collection multiple times. This is significant when we consider all terms of type 2 in conjunction with a set of type 1 terms. Combining all type 2 terms with a set of type 1 terms and factoring tensor products, we get:

$$\begin{aligned}(-1)^{2|x_1|-2} \downarrow x_1 \otimes \Big(\sum_{i+j\leq n} m'_i m'_j(\downarrow x_2 \otimes \downarrow x_3 \otimes \cdots \otimes \downarrow x_n)\Big) \\ = \downarrow x_1 \otimes \big(D^2(\downarrow x_2 \otimes \downarrow x_3 \otimes \cdots \otimes \downarrow x_n)\big) \\ = \downarrow x_1 \otimes 0 \\ = 0\end{aligned}$$

since $D^2 = 0$ when evaluated on $n - 1$ terms.

Hence, all type 2 added together equal 0. All type 3 terms added together equal 0 following a similar argument .

We now consider type 4 terms. Consider an arbitrary element of type 4:

$$m'_i(\downarrow x_1 \otimes \cdots \otimes \downarrow x_i) \otimes \downarrow x_{i+1} \otimes \cdots \otimes \downarrow x_{n-j} \otimes m'_j(\downarrow x_{n-j+1} \otimes \cdots \otimes \downarrow x_n)$$

Consider how this arbitrary element is generated: We begin with

$$m'_i m'_j(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n)$$

We then apply m'_j to the last j terms, which yields:

$$(-1)^{|x_1|+\cdots+|x_{n-j}|-(n-j)} m'_i(\downarrow x_1 \otimes \cdots \otimes \downarrow x_{n-j} \otimes m'_j(\downarrow x_{n-j+1} \otimes \cdots \otimes \downarrow x_n))$$

Finally we apply m'_i to the first i terms:

$$(-1)^{|x_1|+\cdots+|x_{n-j}|-(n-j)} m'_i(\downarrow x_1 \otimes \cdots \otimes \downarrow x_i) \otimes \downarrow x_{i+1} \otimes \cdots \cdots \otimes \downarrow x_{n-j} \otimes m'_j(\downarrow x_{n-j+1} \otimes \cdots \otimes \downarrow x_n) \quad (*)$$

Each of these arbitrary type 4 elements can be paired up with an element generated by $m'_j m'_i$ as follows: Begin with

$$m'_j m'_i(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n)$$

Then apply m'_i to the first i terms:

$$m'_j(m'_i(\downarrow x_1 \otimes \cdots \otimes \downarrow x_i) \otimes \downarrow x_{i+1} \otimes \cdots \otimes \downarrow x_n)$$

Finally, apply m'_j to the last j terms:

$$(-1)^{|x_1|+\cdots+|x_{n-j}|-(n-j)+1} m'_i(\downarrow x_1 \otimes \cdots \otimes \downarrow x_i) \otimes \downarrow x_{i+1} \otimes \cdots \cdots \otimes \downarrow x_{n-j} \otimes m'_j(\downarrow x_{n-j+1} \otimes \cdots \otimes \downarrow x_n) \quad (**)$$

Since these type 4 elements were arbitrary, and $(*) + (**) = 0$, all type 4 terms added together equal 0. Hence, all type 1, 2, 3, and 4 terms yield 0, and so

$$\sum_{i+j \leq n} m'_i m'_j(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) = 0$$

□

Proof of Theorem 2.1. It is clear that each map m_n is of degree $2-n$. To prove that these maps yield an A_∞ structure, one may verify that they satisfy the identity given in definition 1.1. However, this is a rather daunting task, due to the varying signs, s_n, accompanying the m_n maps. To utilize an alternative method of proof, we construct a degree 1 coderivation, D, as described in section 1.

In the context of Theorem 2.1, we may use the definition for m'_k given by Lemma 2.2 to construct D. It then suffices to show that $D^2 = 0$.

We aim to prove $D^2 = 0$ by induction on the number of inputs for D. It is worth first noting that $D = \sum_{k=1}^{\infty} m'_k$, however $D(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) = \sum_{k=1}^{n} m'_k(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n)$ since $m'_k(\downarrow x_1 \otimes \cdots \downarrow \otimes x_n) = 0$ for $k \geq n$.

For $n = 1$, we have $D^2(\downarrow x) = m'_1 m'_1(\downarrow x) = \downarrow m_1^2(x) = 0 \; \forall \, x \in V$.

For $n = 2$, we have

$$\begin{aligned}
D^2(\downarrow x_1, \downarrow x_2) &= m_1' m_1'(\downarrow x_1 \otimes \downarrow x_2) + m_1' m_2'(\downarrow x_1 \otimes \downarrow x_2) \\
&\quad + m_2' m_1'(\downarrow x_1 \otimes \downarrow x_2) + m_2' m_2'(\downarrow x_1 \otimes \downarrow x_2) \\
&= m_1'(m_1'(\downarrow x_1) \otimes \downarrow x_2 - (-1)^{|x_1|} x_1 \otimes m_1'(x_2)) + m_1' m_2'(\downarrow x_1 \otimes \downarrow x_2) \\
&\quad + m_2'(m_1'(\downarrow x_1) \otimes \downarrow x_2 - (-1)^{|x_1|} x_1 \otimes m_1'(x_2)) + 0 \\
&= [m_1' m_1'(\downarrow x_1) \otimes \downarrow x_2 + (-1)^{|x_1|} m_1'(\downarrow x_1) \otimes m_1'(\downarrow x_2)] \\
&\quad - (-1)^{|x_1|}[m_1'(x_1) \otimes m_1'(x_2) - (-1)^{|x_1|} x_1 \otimes m_1' m_1'(x_2)] + m_1' m_2'(\downarrow x_1 \otimes \downarrow x_2) \\
&\quad + m_2'(m_1'(\downarrow x_1) \otimes \downarrow x_2) - (-1)^{|x_1|} m_2'(x_1 \otimes m_1'(x_2)) \\
&= m_1' m_2'(\downarrow x_1 \otimes \downarrow x_2) + m_2'(m_1'(\downarrow x_1) \otimes \downarrow x_2) - (-1)^{|x_1|} m_2'(x_1 \otimes m_1'(x_2)) \\
&= 0 \; \forall \; x_1, \; x_2 \; \in \; V
\end{aligned}$$

Now assume $D^2(\downarrow x_1 \otimes \cdots \downarrow \otimes x_{n-1}) = 0$. We aim to show that $D^2(\downarrow x_1 \otimes \cdots \downarrow \otimes x_n) = 0$: By Lemma 2.4, $D^2(\downarrow x_1 \otimes \cdots \downarrow \otimes x_n) = \sum_{i+j=n+1} m_i' m_j'(\downarrow x_1 \otimes \cdots \downarrow \otimes x_n)$, hence it suffices to show that $\sum_{i+j=n+1} m_i' m_j'(\downarrow x_1 \otimes \cdots \downarrow \otimes x_n) = 0$, $\forall \; x_1 \cdots x_n \; \in \; V$.

It is advantageous to approach this problem from the bottom up, since $x_1 \cdots x_n \; \in \; V$ implies calculating 3^n different combinations of elements. That is, we consider only nontrivial (nonzero) elements in the sum $\sum_{i+j=n+1} m_i' m_j'(\downarrow x_1 \otimes \cdots \downarrow \otimes x_n)$. Now since $i + j = n + 1$, we observe that $m_i' m_j'(\downarrow x_1 \otimes \cdots \downarrow \otimes x_n) \; \in \downarrow V^{\otimes 1}$. Since, by definition, m_i' cannot produce the element $\downarrow v_2$, the seemingly large task of considering nontrivial $m_i' m_j'(\downarrow x_1 \otimes \cdots \downarrow \otimes x_n)$ yields only two possibilities:

$$\begin{aligned}
& m_i' m_j'(\downarrow x_1 \otimes \cdots \downarrow \otimes x_n) = c \downarrow v_1 \\
\text{or} \;\; & m_i' m_j'(\downarrow x_1 \otimes \cdots \downarrow \otimes x_n) = c \downarrow w \text{ for some constant, c.}
\end{aligned}$$

Remark 2.5. *Since production of a $\downarrow v_1$ or $\downarrow w$ relies on the number of v's and w's in the arrangement $\downarrow x_1 \otimes \cdots \downarrow \otimes x_n$, these possibilities are disjoint.*

Therefore if $m_i' m_j'(\downarrow x_1 \otimes \cdots \downarrow \otimes x_n) \neq 0$ for some $i + j = n + 1$, then $\sum_{i+j=n+1} m_i' m_j'(\downarrow x_1 \otimes \cdots \downarrow \otimes x_n)$ contains a collection of $\downarrow v_1$'s or $\downarrow w$'s.

We first consider the manner in which $m_i' m_j'(\downarrow x_1 \otimes \cdots \downarrow \otimes x_n)$ yields a $\downarrow w$:

By defintion of m_n', $\downarrow w$ must be produced by $m_i'(\downarrow v_1 \otimes \downarrow w^{\otimes(i-1)})$ $(*)$. Now since a nonzero m_j' will contribute either the $\downarrow v_1$ or a $\downarrow w$ to the arrangement $\downarrow v_1 \otimes \downarrow w^{\otimes(i-1)}$, the original arrangment $\downarrow x_1 \otimes \cdots \downarrow \otimes x_n$ must contain exactly one more 'v' ($v = v_1$ or v_2), for a total of two v's. It is also worth nothing that $x_1 = v_1$, otherwise $m_i' m_j'(\downarrow x_1 \otimes \cdots \downarrow \otimes x_n) = 0$.

• **Case 1:** $v = v_1$. Then we have $m_i' m_j'(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-2)-k})$, $0 \le k \le n-2$. Now, to produce $(*)$, m_j' must 'catch' (1) both $\downarrow v_1$'s, or (2) only the second $\downarrow v_1$.

(1) We have $m_j'(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-2)-k}) = \downarrow v_1, \ k+2 \le j \le n$.
This yields $m_i'(\downarrow v_1 \otimes \downarrow w^{\otimes (n-j)}) = \downarrow w$. Now since $k+2 \le j \le n$, there are $n-(k+2)+1 = n-k-1$ such terms in $\displaystyle\sum_{i+j=n+1} m_i' m_j'(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-2)-k})$.

(2) We have $(-1)^{|v_1|+k|w|-(k+1)} m_i'\Big(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \Big[m_j'(\downarrow v_1 \otimes \downarrow w^{\otimes (j-1)})\Big] \otimes \downarrow w^{\otimes (n-2)-k-(j-1)}\Big) = -\downarrow w$, $1 \le j \le n-k-1$. Similarly, there are $(n-k-1)-1+1 = n-k-1$ such terms in $\displaystyle\sum_{i+j=n+1} m_i' m_j'(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-2)-k})$.

$$\Rightarrow \sum_{i+j=n+1} m_i' m_j'(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-2)-k}) = (n-k-1)\downarrow w - (n-k-1)\downarrow w = 0.$$

• **Case 2:** $v = v_2$. Then we have $m_i' m_j'(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_2 \otimes \downarrow w^{\otimes (n-2)-k})$, $0 \le k \le n-2$. Similarly, to produce $(*)$, m_j' must 'catch' (1) both $\downarrow v_1$ and $\downarrow v_2$, or (2) only $\downarrow v_2$.

For (1), the only nontrivial way to do this yields:

$$m_{n-k-1}'(m_{k+2}'(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_2) \otimes \downarrow w^{\otimes (n-2)-k}) = \downarrow w$$

and for (2), the only nontrivial way to do this yields:

$$(-1)^{|v_1|+k|w|-(k+1)} m_n'(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes m_1'(\downarrow v_2) \otimes \downarrow w^{\otimes (n-2)-k}) = -\downarrow w$$

$$\Rightarrow \sum_{i+j=n+1} m_i' m_j'(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-2)-k}) = \downarrow w - \downarrow w = 0.$$

In either case, if $m_i' m_j'(\downarrow x_1 \otimes \cdots \downarrow \otimes x_n)$ produces $\downarrow w$'s, then

$$\sum_{i+j=n+1} m_i' m_j'(\downarrow x_1 \otimes \cdots \downarrow \otimes x_n) = 0.$$

We now consider the manner in which $m_i' m_j'(\downarrow x_1 \otimes \cdots \downarrow \otimes x_n)$ yields a $\downarrow v_1$:

By defintion of m'_n, $\downarrow v_1$ must be produced by either $m'_i(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes w^{\otimes(i-2)-k})$ or $m'_i(\downarrow v_1 \otimes \downarrow w^{\otimes(i-2)} \otimes \downarrow v_2)$.

- **Case 1:** $\downarrow v_1$ is produced by $m'_i(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes(i-2)-k})$.

We examine the 4 different possibilities for which m'_j can yield this arrangement:

(i) m'_j produces the first $\downarrow v_1$. (ii) m'_j produces a $\downarrow w$ in $\downarrow w^{\otimes k}$.
(iii) m'_j produces the second $\downarrow v_1$. (iv) m'_j produces a $\downarrow w$ in $\downarrow w^{\otimes(i-2)-k}$.

A key observation to make here is that (i), (ii), (iii), and (iv) imply that the original arrangement $\downarrow x_1 \otimes \cdots \otimes \downarrow x_n$ must contain exactly 3 v's, once again with $x_1 = v_1$. This yields 4 subcases:

Subcase 1: We have $m'_i m'_j(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes l} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes n-k-l-3})$:

(i) m'_j must take the first two $\downarrow v_1$'s. We have:

$$m'_i\Big(\Big[m'_j(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \times \downarrow w^{\otimes j-k-2}) \otimes \downarrow w^{\otimes l-(j-k-2)}\Big] \otimes \downarrow v_1 \otimes \downarrow w^{\otimes n-k-l-3}\Big) = \downarrow v_1$$

Now $k+2 \leq j \leq l+k+2$, so there are $(l+k+2)-(k+2)+1 = l+1$ such terms.

(ii) m'_j must take only the second $\downarrow v_1$. We have:

$$(-1)^{|v_1|+k|w|-(k+1)} m'_i\Big(\downarrow v_1 \otimes w^{\otimes k} \otimes \Big[m'_j(\downarrow v_1 \otimes \downarrow w^{\otimes(j-1)}) \otimes w^{\otimes l-(j-1)}\Big] \otimes \downarrow v_1 \otimes \downarrow w^{\otimes n-k-l-3}\Big) = -\downarrow v_1$$

Now $1 \leq j \leq l+1$, so there are $(l+1)-1+1 = l+1$ such terms.

(iii) m'_j must take the second and third $\downarrow v_1$'s. We have:

$$(-1)^{|v_1|+k|w|-(k+1)} m'_i\Big(\downarrow v_1 \otimes w^{\otimes k} \otimes \Big[m'_j(\downarrow v_1 \otimes \downarrow w^{\otimes l} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes j-l-2})\Big] \otimes \downarrow w^{\otimes n-k-j+1}\Big) = -\downarrow v_1$$

Now $l+2 \leq j \leq n-k-1$, so there are $(n-k-1)-(l+2)+1 = n-k-l-2$ such terms.

(iv) m'_j must take only the third $\downarrow v_1$. We have:

$$(-1)^{2|v_1|+(k+l)|w|-(k+l+2)} m'_i\Big(\downarrow v_1 \otimes w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes l} \otimes \Big[m'_j(\downarrow v_1 \otimes \downarrow w^{\otimes(j-1)}) \otimes \downarrow w^{\otimes n-k-l-j-2}\Big]\Big) = \downarrow v_1$$

Now $1 \leq j \leq n-k-l-2$, so there are $(n-k-l-2)-1+1 = n-k-l-2$ such terms.

$$\Rightarrow \sum_{i+j=n+1} m_i'm_j'(\downarrow v_1\otimes \downarrow w^{\otimes k}\otimes \downarrow v_1\otimes \downarrow w^{\otimes l}\otimes \downarrow v_1\otimes \downarrow w^{\otimes n-k-l-3}) = (l+1)\downarrow v_1 - (l+1)\downarrow v_1 - (n-k-l-2)\downarrow v_1 + (n-k-l-2)\downarrow v_1 = 0.$$

$\circ$ *Subcase 2*: We have $m_i'm_j'(\downarrow v_1\otimes \downarrow w^{\otimes k}\otimes \downarrow v_1\otimes \downarrow w^{\otimes l}\otimes \downarrow v_2\otimes \downarrow w^{\otimes n-k-l-3})$:

By the nature of m_n', it is advantageous to consider whether or not $n-k-l-3=0$:
If $n-k-l-3=0$:
(i) m_j' must take the first two $\downarrow v_1$'s. We have:

$$m_i'\Big(\Big[m_j'(\downarrow v_1\otimes \downarrow w^{\otimes k}\otimes \downarrow v_1\times \downarrow w^{\otimes j-k-2})\otimes \downarrow w^{\otimes l-(j-k-2)}\Big]\otimes \downarrow v_2\Big) = \downarrow v_1$$

Now $k+2\leq j\leq l+k+2$, so there are $(l+k+2)-(k+2)+1=l+1$ such terms.

(ii) m_j' must take only the second $\downarrow v_1$. We have:

$$(-1)^{|v_1|+k|w|-(k+1)}m_i'\Big(\downarrow v_1\otimes w^{\otimes k}\otimes \Big[m_j'(\downarrow v_1\otimes \downarrow w^{\otimes (j-1)})\otimes w^{\otimes l-(j-1)}\Big]\otimes \downarrow v_2\Big) = -\downarrow v_1$$

Now $1\leq j\leq l+1$, so there are $(l+1)-1+1=l+1$ such terms.

(iii) m_j' must take the second $\downarrow v_1$ and $\downarrow v_2$. The only nontrivial way to do this is:

$$(-1)^{|v_1|+k|w|-(k+1)}m_{n-l-1}'\Big(\downarrow v_1\otimes w^{\otimes k}\otimes \Big[m_{l+2}'(\downarrow v_1\otimes \downarrow w^{\otimes l}\otimes \downarrow v_2)\Big]\Big) = -\downarrow v_1$$

(iv) m_j' must take only $\downarrow v_2$. We have:

$$(-1)^{2|v_1|+(k+l)|w|-(k+l+2)}m_n'\Big(\downarrow v_1\otimes w^{\otimes k}\otimes \downarrow v_1\otimes \downarrow w^{\otimes l}\otimes \Big[m_1'(\downarrow v_2)\Big]\Big) = \downarrow v_1$$

$$\Rightarrow \sum_{i+j=n+1} m_i'm_j'(\downarrow v_1\otimes \downarrow w^{\otimes k}\otimes \downarrow v_1\otimes \downarrow w^{\otimes l}\otimes \downarrow v_2) = (l+1)\downarrow v_1 - (l+1)\downarrow v_1 - \downarrow v_1 + \downarrow v_1 = 0.$$

If $n-k-l-3\neq 0$:
(i) and (ii) are trivial.

(iii) m_j' must take the second $\downarrow v_1$ and $\downarrow v_2$. The only nontrivial way to do this is:

$$(-1)^{|v_1|+k|w|-(k+1)}m_{n-l-1}'\Big(\downarrow v_1\otimes w^{\otimes k}\otimes \Big[m_{l+2}'(\downarrow v_1\otimes \downarrow w^{\otimes l}\otimes \downarrow v_2)\Big]\otimes \downarrow w^{\otimes n-k-l-3}\Big) = -\downarrow v_1$$

(iv) m_j' must take only $\downarrow v_2$. We have:

$$(-1)^{2|v_1|+(k+l)|w|-(k+l+2)}m_n'\Big(\downarrow v_1\otimes w^{\otimes k}\otimes \downarrow v_1\otimes \downarrow w^{\otimes l}\otimes \Big[m_1'(\downarrow v_2)\Big]\otimes \downarrow w^{\otimes n-k-l-3}\Big) = \downarrow v_1$$

$$\Rightarrow \sum_{i+j=n+1} m_i'm_j'(\downarrow v_1\otimes \downarrow w^{\otimes k}\otimes \downarrow v_1\otimes \downarrow w^{\otimes l}\otimes \downarrow v_2\otimes \downarrow w^{\otimes n-k-l-3}) = -\downarrow v_1 + \downarrow v_1 = 0.$$

○ *Subcase 3*: We have $m_i'm_j'(\downarrow v_1\otimes \downarrow w^{\otimes k}\otimes \downarrow v_2\otimes \downarrow w^{\otimes l}\otimes \downarrow v_1\otimes \downarrow w^{\otimes n-k-l-3})$:

(i) m_j' must take the first $\downarrow v_1$ and $\downarrow v_2$. The only nontrivial way to do this is:

$$m_{n-k-1}'\Big(\Big[m_{k+2}'(\downarrow v_1\otimes \downarrow w^{\otimes k}\otimes \downarrow v_2)\Big]\otimes \downarrow w^{\otimes l}\otimes \downarrow v_1\otimes \downarrow w^{\otimes n-k-l-3}\Big) =\downarrow v_1$$

(ii) m_j' must take $\downarrow v_2$ only. The only nontrivial way to do this is:

$$(-1)^{|v_1|+k|w|-(k+1)}m_n'\Big(\downarrow v_1 \otimes w^{\otimes k} \otimes \Big[m_1'(\downarrow v_2)\Big]\otimes \downarrow w^{\otimes l}\otimes \downarrow v_1\otimes \downarrow w^{\otimes n-k-l-3}\Big) = - \downarrow v_1$$

Now (iii) and (iv) are trivial.

$$\Rightarrow \sum_{i+j=n+1} m_i'm_j'(\downarrow v_1\otimes \downarrow w^{\otimes k}\otimes \downarrow v_2\otimes \downarrow w^{\otimes l}\otimes \downarrow v_1\otimes \downarrow w^{\otimes n-k-l-3}) =\downarrow v_1- \downarrow v_1 = 0.$$

○ *Subcase 4*: We have $m_i'm_j'(\downarrow v_1\otimes \downarrow w^{\otimes k}\otimes \downarrow v_2\otimes \downarrow w^{\otimes l}\otimes \downarrow v_2\otimes \downarrow w^{\otimes n-k-l-3})$:

If $n-k-l-3 \neq 0$, then this is trivial. Assume $n-k-l-3=0$.

(i) m_j' must take the first $\downarrow v_1$ and first $\downarrow v_2$. The only nontrivial way to do this is:

$$m_{n-k-1}'\Big(\Big[m_{k+2}'(\downarrow v_1\otimes \downarrow w^{\otimes k}\otimes \downarrow v_2)\Big]\otimes \downarrow w^{\otimes l}\otimes \downarrow v_2\Big) =\downarrow v_1$$

(ii) m_j' must take second $\downarrow v_2$ only. The only nontrivial way to do this is:

$$(-1)^{|v_1|+k|w|-(k+1)}m_n'\Big(\downarrow v_1 \otimes w^{\otimes k} \otimes \Big[m_1'(\downarrow v_2)\Big]\otimes \downarrow w^{\otimes l}\otimes \downarrow v_2\Big) = - \downarrow v_1$$

Now (iii) and (iv) are trivial.

$$\Rightarrow \sum_{i+j=n+1} m_i'm_j'(\downarrow v_1\otimes \downarrow w^{\otimes k}\otimes \downarrow v_2\otimes \downarrow w^{\otimes l}\otimes \downarrow v_2\otimes \downarrow w^{\otimes n-k-l-3}) =\downarrow v_1- \downarrow v_1 = 0.$$

Hence, our result holds for case 1.

• **Case 2:** $\downarrow v_1$ is produced by $m_i'(\downarrow v_1\otimes \downarrow w^{\otimes(i-2)}\otimes \downarrow v_2)$.

We examine the 2 different possibilities for which m_j' can yield this arrangement:

(i) m_j' produces the $\downarrow v_1$.
(ii) m_j' produces a $\downarrow w$ in $\downarrow w^{\otimes(i-2)}$.

A similar observation to case 1 can be made here regarding the original arrangement $\downarrow x_1 \otimes \cdots \otimes \downarrow x_n$ containing exactly 3 v's, once again with $x_1 = v_1$. In this case, $x_n = v_2$. This yields 2 subcases:

∘ *Subcase 1*: We have $m'_i m'_j(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes(n-k-3)} \otimes \downarrow v_2)$

(i) m'_j must take both $\downarrow v_1$'s. We have:

$$m'_i\Big(\Big[m'_j(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes j-k-2})\Big] \otimes \downarrow w^{\otimes n-j-1} \otimes \downarrow v_2\Big) = \downarrow v_1$$

Now $k+2 \leq j \leq n-1$, so there are $(n-1)-(k+2)+1 = n-k-2$ such terms.

(ii) m'_j must take the second $\downarrow v_1$ only. We have:

$$(-1)^{|v_1|+k|w|-(k+1)} m'_i\Big(\downarrow v_1 \otimes w^{\otimes k} \otimes \Big[m'_j(\downarrow v_1 \otimes \downarrow w^{\otimes j-1}) \otimes \downarrow w^{\otimes n-k-j-2}\Big] \otimes \downarrow v_2\Big) = -\downarrow v_1$$

Now $1 \leq j \leq n-k-2$, so there are $(n-k-2)-(1)+1 = n-k-2$ such terms.
This implies that

$\sum_{i+j=n+1} m'_i m'_j(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes(n-k-3)} \otimes \downarrow v_2) = (n-k-2)\downarrow v_1 - (n-k-2)\downarrow v_1 = 0.$

∘ *Subcase 2*: We have $m'_i m'_j(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_2 \otimes \downarrow w^{\otimes(n-k-3)} \otimes \downarrow v_2)$

(i) m'_j must take $\downarrow v_1$ and the first $\downarrow v_2$. The only nontrivial way to do this is:

$$m'_{n-k-1}\Big(\Big[m'_{k+2}(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_2)\Big] \otimes \downarrow w^{\otimes n-k-3} \otimes \downarrow v_2\Big) = \downarrow v_1$$

(ii) m'_j must take second $\downarrow v_2$ only. The only nontrivial way to do this is:

$$(-1)^{|v_1|+k|w|-(k+1)} m'_n\Big(\downarrow v_1 \otimes w^{\otimes k} \otimes \Big[m'_1(\downarrow v_2)\Big] \otimes \downarrow w^{\otimes n-k-3} \otimes \downarrow v_2\Big) = -\downarrow v_1$$

Now (iii) and (iv) are trivial.

$$\Rightarrow \sum_{i+j=n+1} m'_i m'_j(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_2 \otimes \downarrow w^{\otimes(n-k-3)} \otimes \downarrow v_2) = \downarrow v_1 - \downarrow v_1 = 0.$$

Hence, our result holds for case 2.

So $\displaystyle\sum_{i+j=n+1} m'_i m'_j(\downarrow x_1 \otimes \cdots \downarrow \otimes x_n) = 0, \ \forall \ x_1 \cdots x_n \ \in \ V.$

Thus $D^2(\downarrow x_1 \otimes \cdots \downarrow \otimes x_n) = 0$

By induction, $D^2 = 0$ on any number of inputs.

Hence the preceding maps m_n on the graded vector space V form an A_∞ algebra. □

3. Induced L_∞ Algebra

The A_∞ algebra structure on $V = V_0 \oplus V_1$ that was constructed in this note may be skew symmetrized to yield an L_∞ algebra structure on V; see [5] for details. This L_∞ algebra will thus join the collection of previously defined such structures on V. The relationship among these algebras will be a topic for future research.

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