OCHA:

Examples & Related Structures

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Fix a ground field k, $\mathrm{char}(k) = 0$.

Outline:

1) Introduction to OCHA

2) Examples:
OCHA from A_∞-extensions

3) Related Structures
3.1) $\mathfrak{g}$-algebras (or Leibniz Pairs)
3.2) Swiss-Cheese Operad

4) OCHA on Singular Chains of relative 2-loop Spaces:
$A = C_*(\Omega^2(X, A))$; $L = C_*(\Omega^2 X)$

Work in Progress

More realistic outline:

1) Introduction to OCHA

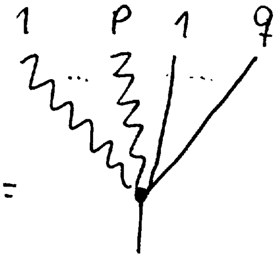
2) L_∞-algebras from commutators of A_∞-algebras (Lada-Markl)

3) A_∞-extensions

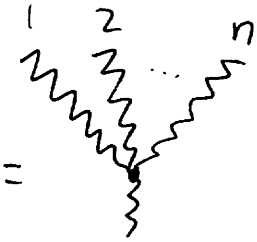
4) OCHA from commutators and shufflings of A_∞-extensions

1) What is ... an OCHA ?

it is a pair of DG-spaces (L, A) endowed with multilinear operations

$$\eta_{p,q} : L^{\wedge p} \otimes A^{\otimes q} \longrightarrow A \,, \quad \eta_{p,q} =$$ 

$p+q \geq 1$ and

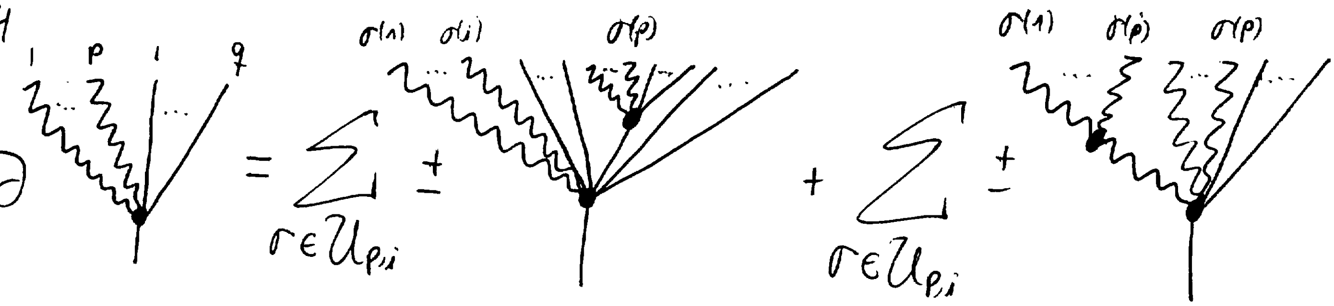
$$l_n : L^{\wedge n} \longrightarrow L \,, \quad l_n =$$ 

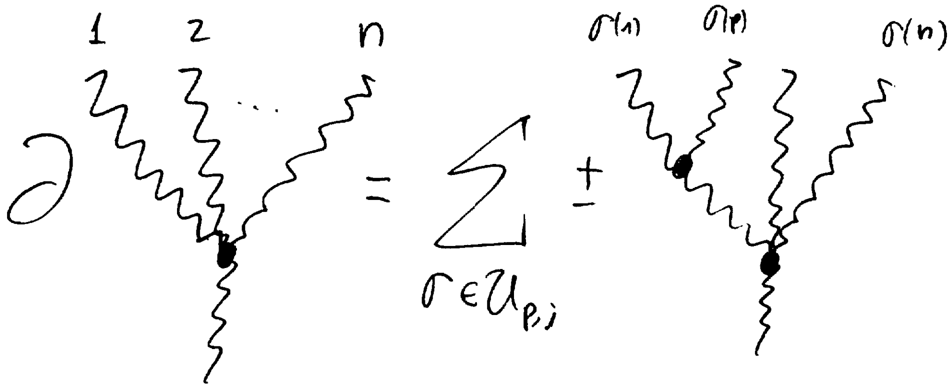
$n \geq 1$

$\eta_{0,1} = d_A \quad l_1 = d_L$

partially planar trees

Satisfying Certain Relations:

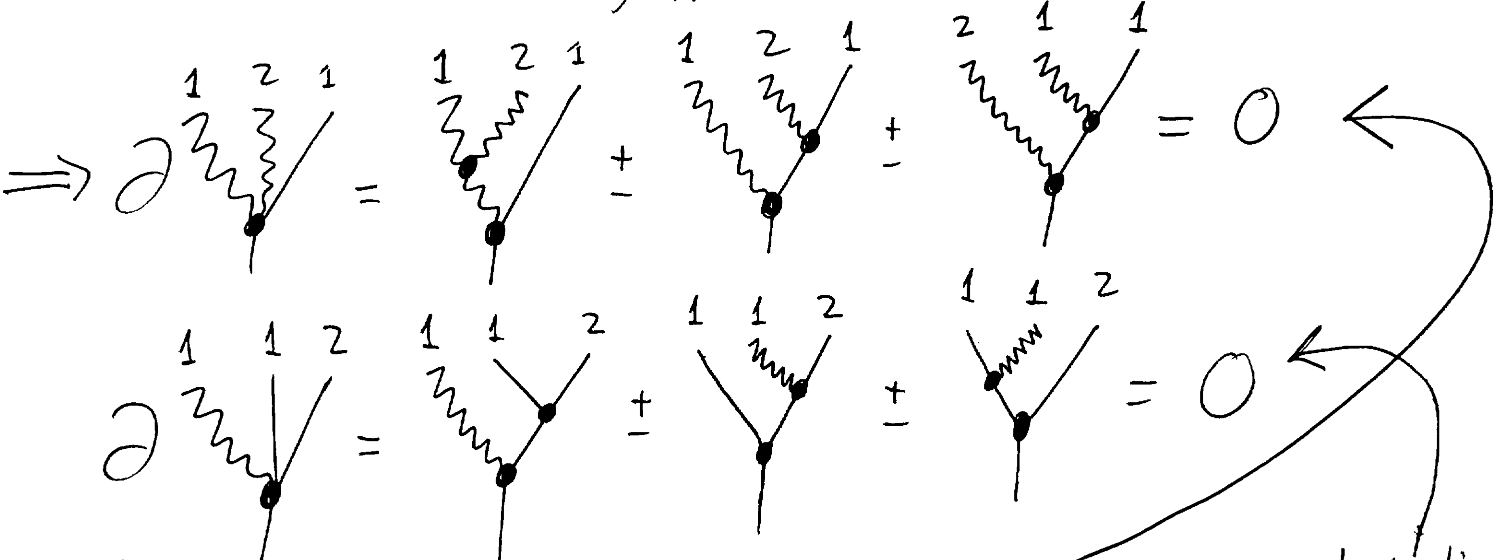




Example (Lie algebra action by derivations):

Assume $n_{p,0} = 0 \ \forall p \geq 1$, $n_{p,q} = 0 \ \forall p+q > 2$

and $l_n = 0$ for all $n > 2$



 $: L \otimes A \to A$ is a Lie algebra action by derivations

Kajiura & Stasheff noticed that:

$$\eta_{p,q} : L^{\wedge p} \otimes A^{q} \longrightarrow A$$

$$l_n : L^{\wedge n} \longrightarrow L$$

Can be lifted to $\bar{\eta}_{p,q}$, $\bar{l}_n : S(L) \otimes T(A) \circlearrowleft$

So $D = \sum \bar{\eta}_{p,q} + \sum \bar{l}_n \in \mathrm{Coder}\big(S(L) \otimes T(A)\big)$

OCHA Relations: $[D, D] = 0$

Theorem (Hoefel, 06, arxiv)

Any coderivation on $\mathrm{Coder}(S(L) \otimes T(A))$ is of the form $\sum\limits_{p+q \geq 1} \bar{\eta}_{p,q} + \sum\limits_{n \geq 1} \bar{l}_n$.

So, an OCHA is simply:

$$D \in \mathrm{Coder}^1(S(L) \otimes T(A)), \quad [D, D] = 0.$$

Question:

How can we get an OCHA out of an A_∞-algebra?

Lada - Markl (94 Comm. Alg.):

Symmetrization:

$$\Psi : S(L) \longrightarrow T(L)$$

$$x_1 \wedge \dots \wedge x_n \longmapsto \sum_{\sigma \in S_n} x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)}$$

If $D \in \mathrm{Coder}(T(L))$, $[D, D] = 0$ is an A_∞-algebra on L, then

$$\xi = \Psi^{-1} D \Psi \in \mathrm{Coder}(S(L))$$

$$[\xi, \xi] = 0$$

defines an L_∞-structure on L

- Lada-Markl introduced the Universal Enveloping A_∞ Algebra of an L_∞-algebra

A_∞-extensions: (New definition?)

Let A and B be A_∞-algebras.

We say that a DG-space E is an <u>A_∞-extension</u> of B by A if E has an A_∞-algebra structure and fits into an exact sequence of A_∞-algebras:

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

A_∞-morphisms

An A_∞-extension

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

is equivalent to the following data:
up to isomorphism:

1) $E = A \oplus B$ as k-vector spaces

2) A is an A_∞-ideal of E

meaning that $m_k(\cdots, a, \cdots) \in A$ if $a \in A$

in particular $\underline{d_E a \in A}$ $\forall\, a \in A$

where $M = d + m_2 + m_3 + \cdots$ is A_∞-structure of E.

So $M : T(A \oplus B) \longrightarrow A \oplus B$ is decomposed

into $N : T(A \oplus B) \longrightarrow A$

and

$P : T(B) \longrightarrow B$

(10)

$P : T(B) \longrightarrow B$ defines an A_∞-algebra structure on B.

Let L be the L_∞-algebra obtained by symmetrization of B

$$\sum \bar{l}_n = P \circ \Psi : S(L) \longrightarrow L$$

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and consider the coalgebra morphism:

$$\Theta : T(A) \otimes S(L) \longrightarrow T(A \oplus L)$$

$$(a_1 \otimes \cdots \otimes a_q) \otimes (X_1, \dots, X_p) \mapsto Sh\Big(a_1 \otimes \cdots \otimes a_q \,\Big|\, \sum_{\sigma \in S_n} \pm X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(p)}\Big)$$

Where Sh denotes the shuffle product

Then we finally have:

$$\sum \bar{n}_{p,q} = N \circ \Theta : T(A) \otimes S(L) \longrightarrow A$$

and

$$\sum \bar{l}_n = P \circ \Psi : S(L) \longrightarrow L$$

<u>Theorem</u> (Hoefel, to appear)

The above maps $n_{p,q} : A^{\otimes p} \otimes L^{\wedge q} \longrightarrow A$ and $l_n : L^{\wedge n} \longrightarrow L$ define an OCHA structure on the pair (A, L).

<u>Proof</u>

$$\underline{\Theta^{-1}} \circ \underline{M \circ \Theta} \in \mathrm{Coder}^1(T(A) \otimes S(L))$$

Just Aplly previous theorem. ∎