Explicit higher Hochschild complexes

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Outline:

1. Hochschild complex for A
2. Examples of higher Hochschild complexes for commutative A
3. General definition + Chen iterated integral map
4. Applications
   - Surface product
   - Computations (HKR, ...)

Joint work with G. Ginot, M. Zeinalian

1. Hochschild complex

$(A, d, \cdot)$ diff. graded assoc. alg.

$$CH_\bullet(A) := \prod_{n \geq 0} A^{\otimes n+1}$$

Differential:

$$\begin{aligned}
&b(a_0 \otimes a_1 \otimes \ldots \otimes a_n) \\
&= (a_0 \cdot a_1) \otimes a_2 \otimes \ldots \otimes a_n \\
&\quad \pm a_0 \otimes (a_1 \cdot a_2) \otimes a_3 \otimes \ldots \otimes a_n \\
&\quad \pm \ldots \\
&\quad \pm a_0 \otimes a_1 \otimes \ldots \otimes (a_{n-1} \cdot a_n) \\
&\quad \pm (a_n \cdot a_0) \otimes a_1 \otimes \ldots \otimes a_{n-1}
\end{aligned}$$

$$\begin{aligned}
&d(a_0 \otimes a_1 \otimes \ldots \otimes a_n) \\
&= \sum_i \pm a_0 \otimes \ldots \otimes da_i \otimes \ldots \otimes a_n
\end{aligned}$$

$$b^2 = d^2 = db + bd = 0$$

$a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \quad a_8 \quad a_9$

b: $a_2, a_3 \longmapsto a_2 \cdot a_3$

d: $a_6 \longmapsto da_6$

$\Rightarrow$ circle

$\mathcal{L}M = \mathrm{Map}(S^1, M)$ free loop space

$$\begin{array}{ccc} \mathcal{L}M \times \Delta^k & \xrightarrow{ev} & M^{k+1} \\ \Big\downarrow {\scriptstyle \int_{\Delta^k}} & & \\ \mathcal{L}M & & \end{array}$$

Let $A = \Omega_{DR}(M)$ De Rham forms

<u>Thm:</u> (Chen)

$$\Omega(M)^{\otimes k+1} \xrightarrow{ev^*} \Omega(\mathcal{L}M \times \Delta^k) \xrightarrow{\int_{\Delta^k}} \Omega(\mathcal{L}M)$$

induces $HH_\cdot(A) \xrightarrow{\cong} H^\cdot(\mathcal{L}M)$

for M simply connected.

2. Examples of higher Hochschild

(a) The torus $\mathbb{T}$

$(A, d, \cdot)$ diff. grad. assoc. <u>commut.</u> algebra

$$CH_\bullet^{\mathbb{T}}(A) := \prod_{n \geq 0} A^{\otimes (n+1)^2}$$

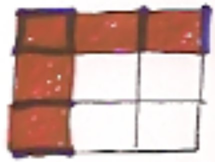
$$\begin{matrix} a_{00} \otimes a_{01} \otimes a_{02} \otimes \cdots \otimes a_{0n} \\ \otimes a_{10} \otimes a_{11} \otimes a_{12} \otimes \cdots \otimes a_{1n} \\ \vdots \qquad \vdots \qquad \ddots \qquad \vdots \\ \otimes a_{n0} \otimes a_{n1} \otimes a_{n2} \otimes \cdots \otimes a_{nn} \end{matrix}$$

Differential: $(b+d)^2 = 0$

$$d(\ldots) = \sum_{i,j} \ldots \otimes d(a_{ij}) \otimes \ldots$$

$$b\begin{pmatrix} a_{00} \otimes a_{01} \otimes a_{02} \otimes a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} \rightleftarrows$$

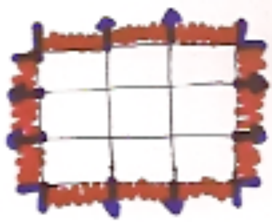
$$\rightleftarrows \begin{array}{ccc} a_{00}\cdot a_{01}\cdot a_{10}\cdot a_{11} \otimes & a_{02}\cdot a_{12} \otimes & a_{03}\cdot a_{13} \\ a_{20}\cdot a_{21} & a_{22} & a_{23} \\ a_{30}\cdot a_{31} & a_{32} & a_{33} \end{array}$$



$$\rightleftarrows \begin{array}{ccc} a_{00} \otimes & a_{01}\cdot a_{02} \otimes & a_{03} \\ a_{10}\cdot a_{20} & a_{11}\cdot a_{12}\cdot a_{21}\cdot a_{22} & a_{13}\cdot a_{23} \\ a_{30} & a_{31}\cdot a_{32} & \end{array}$$

$$\rightleftarrows \begin{array}{ccc} a_{00} \otimes & a_{01} \otimes & a_{02}\cdot a_{03} \\ a_{10} & a_{11} & a_{12}\cdot a_{13} \\ a_{20}\cdot a_{30} & a_{21}\cdot a_{31} & a_{22}\cdot a_{23}\cdot a_{32}\cdot a_{33} \end{array}$$

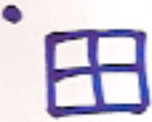
$$\rightleftarrows \begin{array}{ccc} a_{00}\cdot a_{03}\cdot a_{30}\cdot a_{33} \otimes & a_{01}\cdot a_{31} \otimes & a_{02}\cdot a_{32} \\ a_{10}\cdot a_{13} & a_{11} & a_{12} \\ a_{20}\cdot a_{23} & a_{21} & a_{22} \end{array}$$



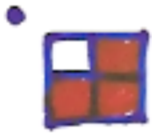
(b) The 2-sphere S^2

$$CH_\bullet^{S^2}(A) := \prod_{n \geq 0} A^{\otimes n^2+1}$$

$$\begin{array}{lllll}
a_{00} & & & & \\
\otimes\, a_{11} & \otimes\, a_{12} & \otimes & \cdots & \otimes\, a_{1n} \\
\otimes\, a_{21} & a_{22} & & \cdots & a_{2n} \\
\vdots & \vdots & & \ddots & \vdots \\
a_{n1} & a_{n2} & & \cdots & a_{nn}
\end{array}$$

$b\Big($  $\Big) =$ 

$\pm$ 

$\pm$ 

$\pm$ 

$d\Big($  $\Big) =$

$$= \sum_{i,j} \cdots \otimes d(a_{ij}) \otimes \cdots$$

$$(b+d)^2 = 0$$

Let M be a 2-connected mfld.

Let $A = \Omega_{DR}(M)$ De Rham forms.

Let $Y = \mathbb{T}$ or S^2,

$M^Y = \mathrm{Maps}(Y, M)$.

$$\begin{tikzcd} M^Y \times \Delta^k \arrow[r, "ev"] \arrow[d, "\int_{\Delta^k}"'] & M^{(k+1)^2} \\ M^Y & \end{tikzcd}$$

(or M^{k^2+1})

<u>Thm:</u>

(a) $\Omega(M)^{\otimes (k+1)^2} \longrightarrow \Omega(M^{\mathbb{T}} \times \Delta^k) \to \Omega(M^{\mathbb{T}})$

induces isom. $HH_\bullet^{\mathbb{T}}(\Omega M) \longrightarrow H^\bullet(M^{\mathbb{T}})$

(b) $\Omega(M)^{\otimes k^2+1} \longrightarrow \Omega(M^{S^2} \times \Delta^k) \to \Omega(M^{S^2})$

induces isom. $HH_\bullet^{S^2}(\Omega M) \longrightarrow H^\bullet(M^{S^2})$.

3. General definition

[Pirashvili, 2000]

Let Δ : category
objects: sets $\{0,\dots,n\} = \underline{n}$
morph.: non-decreasing set maps
$f(i) \le f(j)$ for $i \le j$.

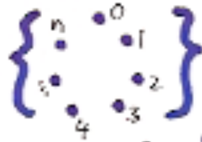
Let Sets: category of sets.
Let dgVect: category of diff. graded vector spaces/k.

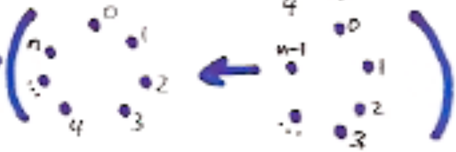
<u>Definition:</u>
A <u>simplicial set</u> is a contravariant functor $\Delta \longrightarrow$ Sets.

A <u>simplicial dg v.s.</u> is a contravariant functor $\Delta \longrightarrow$ dgVect.

Examples of simplicial sets:

circle S^1: $\quad \Delta \to \mathrm{Sets}$

$\underline{n} \mapsto$ 

$(f : \underline{n} \to \underline{n+1}) \mapsto$ 

torus $\mathbb{T}$: $\quad \Delta \longrightarrow \mathrm{Sets}$

$\underline{n} \mapsto$ 

sphere S^2: $\quad \Delta \longrightarrow \mathrm{Sets}$

$\underline{n} \mapsto$ 

Goal:

Start from a simplicial set Y

$$\Delta \xrightarrow{Y} \text{Sets}$$

and produce a simplicial dg v.s.

$$\Delta \longrightarrow \text{dgVect}$$

Solution:

Use a functor $\text{Sets} \to \text{dgVect}$ and compose with Y.

Note:

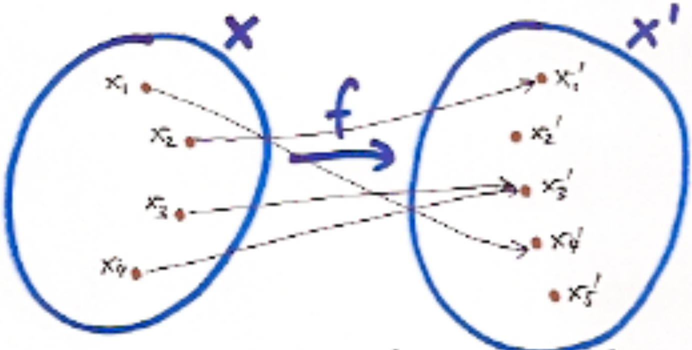
Every $(A, d, \cdot)$ d.g. assoc. commut. alg. induces a functor $\text{Sets} \xrightarrow{A} \text{dgVect}$ by setting:

$$(A, d, \cdot) : \quad \mathrm{Sets} \longrightarrow \mathrm{dgVect}$$

$$X \longmapsto A^{\otimes X}$$

$$(f : X \to X') \longmapsto \left(A^{\otimes X} \xrightarrow{f_*} A^{\otimes X'}\right)$$

where f_* is given by:



$$f_* : a_1 \otimes \ldots \otimes a_n \mapsto b_1 \otimes \ldots \otimes b_{n'}$$

where $b_j = \prod\limits_{\substack{i \text{ s.t.} \\ f(i)=j}} a_i$

$\left(\text{or} \quad b_j = 1 \quad , \text{if } \nexists\, i : f(i) = j\right)$

Example:

$$a_{x_1} \otimes a_{x_2} \otimes a_{x_3} \otimes a_{x_4} \longmapsto$$

$$a_{x_2} \otimes 1 \otimes (a_{x_3} \cdot a_{x_4}) \otimes a_{x_1} \otimes 1 .$$

Conclusion:

Simplicial set $Y: \Delta \to \text{Sets}$

Commutative dga $(A, d, \cdot, 1)$

$$A: \text{Sets} \to \text{dgVect}$$

induces

$$\boxed{CH_\bullet^Y(A): \Delta \to \text{Sets} \to \text{dgVect}}$$

The total space

$$\boxed{CH_\bullet^Y(A) = \prod_{n \geq 0} A^{\otimes |Y(\underline{n})|}}$$

has differential $(b+d)^2 = 0$.

$$b = \sum \pm d_i$$

where $d_i = (Y(d_i))_*$ comes from the i^{th} face operator;

d = extension of d on A to $A^{\otimes \cdots}$ as a derivation.

$$HH_\bullet^Y(A) := H_\bullet(CH_\bullet^Y(A), b+d)$$

is called <u>Hochschild homology</u> of A with respect to Y.

<u>Chen iterated integrals:</u>

$$M^Y := \mathrm{Maps}(|Y|, M)$$

$$\begin{tikzcd} M^Y \times \Delta^k \arrow[r, "ev"] \arrow[d, "\int_{\Delta^k}"'] & M^{|Y(\underline{k})|} \\ M^Y & \end{tikzcd}$$

$$\Rightarrow \Omega(M)^{\otimes |Y(\underline{k})|} \xrightarrow{ev^*} \Omega(M^Y \times \Delta^k) \xrightarrow{\int_{\Delta^k}} \Omega(M^Y)$$

induces an isomorphism

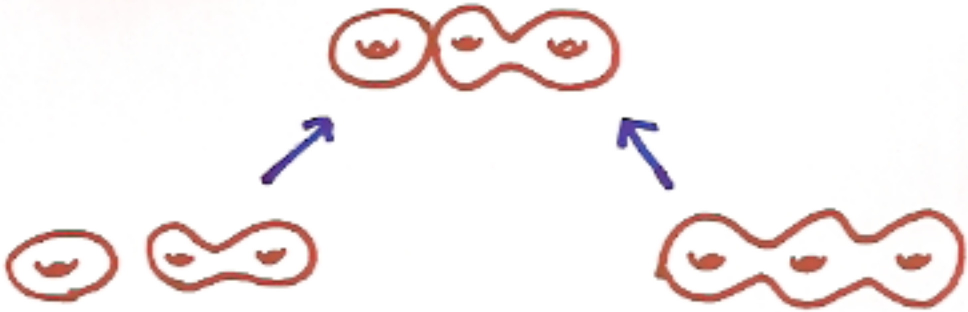
$$\boxed{HH_\bullet^Y(\Omega M) \longrightarrow H^\bullet(M^Y)},$$

assuming M is at least $\dim(Y)$-connected.

[Ginot - T. - Zeinalian]

4. Applications

(a) Surface Product



$$\Rightarrow M^{\Sigma^g} \times M^{\Sigma^h} \xleftarrow{\rho_{in}} M^{\Sigma^g \vee \Sigma^h} \xrightarrow{\rho_{out}} M^{\Sigma^{g+h}}$$

$$\begin{tikzcd}
M^{\Sigma^g \vee \Sigma^h} \arrow[r, "\rho_{in}"] \arrow[d] & M^{\Sigma^g} \times M^{\Sigma^h} \arrow[d] \\
M \arrow[r, "\text{diagonal}"] & M \times M
\end{tikzcd}$$

ρ_{in}: embedding with finite codimension

$\Rightarrow$ Umkehr map for S_{in}:

$$
\begin{aligned}
& H_\bullet(M^{\Sigma_g}) \otimes H_\bullet(M^{\Sigma_h}) \\
& \cong H_\bullet(M^{\Sigma_g} \times M^{\Sigma_h}) \\
& \xrightarrow[\text{Thom collapse}]{} H_\bullet(\mathrm{Thom}(M^{\Sigma_g \vee \Sigma_h})) \\
& \xrightarrow[\text{Thom isomorphism}]{} H_{\bullet - \dim M}(M^{\Sigma_g \vee \Sigma_h}) \\
& \xrightarrow[(S_{out})_*]{} H_{\bullet - \dim M}(M^{\Sigma_{g+h}})
\end{aligned}
$$

Call this the <u>surface product.</u>

Model this with

$H_\bullet(M^Y) \cong HH^\bullet_Y(\Omega M)$, where

$CH^\bullet_Y(\Omega M) := \mathrm{Hom}_{\Omega M}(CH_\bullet^Y(\Omega M), \Omega M)$