



Yale University  
Department of Mathematics

Field Equations from  
Homotopy Algebras of CFT

Anton M. Zeitlin

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- Motivation
- Reminder of Lian-Zuckerman (LZ) homotopy algebras
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## Motivation

String theory:

2d Conformal Field Theory  $\longrightarrow$

D-dimensional Quantum Field Theory

Linear classical field equations (Maxwell, Linearized Einstein) and their symmetries:

$$Q\Phi = 0, \quad \Phi \longrightarrow \Phi + Q\lambda$$

$Q$  is a semi-infinite cohomology operator for the Virasoro algebra.

What about nonlinear equations?

String Field Theory:

$$Q\Phi + \sum_n \mu_n(\Phi, \dots, \Phi) = 0$$
$$Q, \mu_n : A_\infty / L_\infty$$

The description of  $\mu_n$  is too complicated.  
Something more explicit?

# Natural mathematical tool to study 2d CFT: VOA (vertex operator algebra)

**B.H. Lian, G.J. Zuckerman**

”New Perspectives on the  
BRST-algebraic Structure of  
String Theory”

Commun.Math.Phys.154 (1993)  
613

## LZ homotopy algebra (reminder)

Let  $V$  be the VOA.  $T(z)$  is a Virasoro element

Semi-infinite complex :  $C^* = V \otimes \Lambda$

$\Lambda$  - VOA of “conformal ghosts”:

$$b(z)c(w) \sim \frac{1}{z-w}$$

$$Q = \oint dz c(z)T(z) + :c\partial cb:(z)$$

semi-infinite cohomology (BRST) operator.

Let  $a(z)$  be a vertex operator for  $a \in V \otimes \Lambda$

Operations:

$$\mu(a_1, a_2) \equiv P_0 a_1(\varepsilon) a_2$$

$$\{a_1, a_2\} \equiv (-1)^{|a_1|} \oint dz (b_{-1} a_1)(z) a_2$$

where  $P_0$  is the projection on  $\varepsilon$  - independent component.

## Proposition 1

- (i)  $\mu(a_1, a_2) - (-1)^{|a_1||a_2|}\mu(a_2, a_1) =$   
 $Qm(a_1, a_2) + m(Qa_1, a_2) + (-1)^{|a_1|}m(a_1, Qa_2)$
- (ii)  $\mu(\mu(a_1, a_2), a_3) - \mu(a_1, \mu(a_2, a_3)) =$   
 $Qn(a_1, a_2, a_3) + n(Qa_1, a_2, a_3) + (-1)^{|a_1|}n(a_1, Qa_2, a_3)$   
 $+(-1)^{|a_1|+|a_2|}n(a_1, a_2, Qa_3),$

## Lemma

$$(-1)^{|a_2|}\{a_1, a_2\} = b_0\mu(a_1, a_2) - \mu(b_0a_1, a_2) - (-1)^{|a_2|}\mu(a_1, b_0a_2)$$

## Proposition 2

- (i)  $\{a_1, a_2\} + (-1)^{(|a_1|-1)(|a_2|-1)}\{a_2, a_1\} =$   
 $(-1)^{|a_1|-1}(Qm'(a_1, a_2) + m'(Qa_1, a_2)$   
 $-(-1)^{|a_2|}m'(a_1, Qa_2))$
- (ii)  $\{\{a_1, a_2\}, a_3\} - \{a_1, \{a_2, a_3\}\}$   
 $+(-1)^{(|a_1|-1)(|a_2|-1)}\{a_2, \{a_1, a_3\}\} = 0$
- (iii)  $\{a_1, \mu(a_2, a_3)\} = \mu(\{a_1, a_2\}, a_3)$   
 $+(-1)^{(|a_1|-1)||a_2|}\mu(a_2, \{a_1, a_3\})$
- (iv)  $\{\mu(a_1, a_2), a_3\} - \mu(a_1, \{a_2, a_3\})$   
 $-(-1)^{(|a_3|-1)|a_2|}\mu(\{a_1, a_3\}, a_2) =$   
 $(-1)^{|a_1|+|a_2|-1}(Qn''(a_1, a_2, a_3) - n''(Qa_1, a_2, a_3) -$   
 $(-1)^{|a_2|}n''(a_1, Qa_2, a_3) - (-1)^{|a_1|+|a_2|}n''(a_1, a_2, Qa_3))$

Therefore  $V \otimes \Lambda$  carry a structure of homotopy Gerstenhaber algebra.

One can generalize further:

$$\begin{aligned} \mathbf{C}^* &= C^* \otimes \bar{C}^* \\ \mu^{\text{ext}}(a_1, a_2) &= P_0 a_1(\epsilon) a_2, \quad \epsilon \notin \mathbb{R} \\ \{a_1, a_2\}^{\text{ext}} &= P_0 \oint_{C_{\epsilon,0}} a_1^{(1)} a_2, \end{aligned}$$

where  $a^{(1)} = dz(b_{-1}a)(z) + d\bar{z}(\bar{b}_{-1}a)(z)$ .

$\mu^{\text{ext}}, \{\cdot, \cdot\}^{\text{ext}}$  also satisfy the relations of homotopy Gerstenhaber algebra w.r.t. operator  $Q = Q + \bar{Q}$

Conjecture (L-Z) Operations  $\mu, \{\cdot, \cdot\}$  and  $\mu^{\text{ext}}, \{\cdot, \cdot\}^{\text{ext}}$  can be extended to the structure of  $G_\infty$  algebra.

Attempts to prove:

**Kimura, Voronov, Zuckerman**

“Homotopy Gerstenhaber algebra and topological field theory”

q-alg/9602009

**Galvez, Gorbounov, Tonks**

“Homotopy Gerstenhaber Structures and Vertex Algebras”

math/0611231

What is the physical meaning of the associated Maurer-Cartan equations?

$$Q\Phi + \mu(\Phi, \Phi) + \sum_{n=3}^{\infty} \mu_n(\Phi, \dots, \Phi) = 0$$
$$Q\Psi + \frac{1}{2}\{\Psi, \Psi\} + \sum_{n=3}^{\infty} \frac{1}{n!}\{\Psi, \dots, \Psi\}_n = 0$$

We will show that

- a) they lead to nonlinear field equations and their symmetries
- b) can give rise to an “algebraic” definition of  $\beta$ -function for the perturbed CFT ( $\sigma$ -models in particular)

## Perturbed CFTs

VOA  $\longrightarrow$  CFT with some action  $S_0$ .

Perturbations:  $S_0 \longrightarrow S = S_0 + V$

$$V = \int_{\Sigma} \Phi^{(2)}, \quad \Phi^{(2)} = dz \wedge d\bar{z} A(z), \quad A \in C^* \otimes \bar{C}^*$$

In general, the perturbed theory is not a CFT.  
Renormalization theory gives the condition for the theory to be conformal:

$$\beta(\Phi) = 0$$

Expand:  $\Phi^{(2)} = t\Phi_1^{(2)} + t^2\Phi_2^{(2)} + \dots$

$$\beta_1(V) = 0 \iff Q\Phi_1^{(0)} = 0$$

$$\beta_2(V) = 0 \iff Q\Phi_2^{(0)} + \frac{1}{4\pi i} P_0 \oint_{C_{\varepsilon,0}} \Phi_1^{(1)} \Phi_1^{(0)} = 0$$

where  $\Phi_i^{(2)} = dz \wedge d\bar{z} [b_{-1}, [\bar{b}_{-1}, \Phi_i^{(0)}]]$

**A. Sen** (89,90);

**A.M.Z.**

“BRST, Generalized Maurer-Cartan Equations and CFT”,

Nucl.Phys.B 794 (2006) 370,

“Formal Maurer-Cartan Structures:

From CFT to Classical Field Equations”,

JHEP0709:098(2007)



Therefore one can replace  $\beta(\Phi)$  by

$$\hat{\beta}(\Phi) = Q\Phi + \frac{1}{2}\{\Phi, \Phi\} + \sum_{n=3}^{\infty} \frac{1}{n!}\{\Phi, \dots, \Phi\}_n$$

where  $\Phi^{(2)} = dz \wedge d\bar{z}[b_{-1}, [\bar{b}_{-1}, \Phi]]$

One can think of  $\hat{\beta}(\Phi)$  as an “algebraic” definition of  $\beta$ -function.

The same applies to  $\beta$ -functions for the theories with boundary perturbations:

$$S_0 \longrightarrow S = S_0 + \int_{\partial\Sigma} \Phi^{(1)}$$

In this case ”algebraic”  $\beta$ -function is:

$$\beta(\Phi) = Q\Phi + \mu(\Phi, \Phi) + \sum_{n=3}^{\infty} \mu_n(\Phi, \dots, \Phi)$$

where  $\Phi^{(1)} = dz[b_{-1}, \Phi] + d\bar{z}[\bar{b}_{-1}, \Phi]$

## $\sigma$ -models

$$S = \frac{1}{2\pi h} \int_{\Sigma} d^2 z (G_{\mu\nu}(X) + B_{\mu\nu}(X)) \partial X^{\mu} \bar{\partial} X^{\nu} \\ + \frac{1}{2\pi} \int R^{(2)} \phi(X) + \int_{\partial\Sigma} A_{\mu}(X) dX^{\mu} \\ \beta(G, B, \phi, A) = \sum_{n=1}^{\infty} h^n \beta_n(G, B, \phi, A)$$

$\beta_1 = 0 \iff$  **Einstein-Yang-Mills equations**  
 $\beta_n = 0$  ( $n > 1$ ) gives equations with higher derivatives

### Complications:

i)  $S = S_0 + V$  destroys the geometric background:

$$G_{\mu\nu} = \eta_{\mu\nu} + t g_{\mu\nu}^{(1)} + t^2 g_{\mu\nu}^{(2)} + \dots$$

ii)

$$S_0 = \frac{1}{2\pi h} \int_{\Sigma} d^2 z \eta_{\mu\nu} \partial X^{\mu} \partial X^{\nu}$$

$$X^{\mu}(z) X^{\nu}(w) \sim h \eta^{\mu\nu} \ln |z - w|^2$$

**logarithmic VOA**

Nevertheless, if we neglect logarithms Maurer-Cartan equation

$$Q\Phi + \frac{1}{2}\{\Phi, \Phi\} + \dots = 0, \quad \Phi(G, B, \Phi)$$

reproduces Einstein equations ( $H = dB$ ):

$$\begin{aligned} R_{\mu\nu} &= -\frac{1}{4}H_{\mu\lambda\rho}H_{\nu}^{\lambda\rho} + 2\nabla_{\mu}\nabla_{\nu}\phi, \\ \nabla_{\mu}H^{\mu\nu\rho} - 2(\nabla_{\lambda}\phi)H^{\lambda\nu\rho} &= 0, \\ 4(\nabla_{\mu}\phi)^2 - 4\nabla_{\mu}\nabla^{\mu}\phi + R + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} &= 0 \end{aligned}$$

and their symmetries up to the second order in  $t$ .

**A.M.Z.**

“Formal Maurer-Cartan Structures:  
from CFT to Classical Field Equations”  
JHEP0712:098(2007)

We need another formulation:

$$\begin{aligned} S = \frac{1}{2\pi h} \int_{\Sigma} d^2z & (p_i \bar{\partial} X^i + p_{\bar{i}} \partial X^{\bar{i}} - \\ & (g^{i\bar{j}} p_i p_{\bar{j}} + \mu_{\bar{j}}^i p_i \bar{\partial} X^{\bar{j}} + \\ & \mu_{\bar{j}}^{\bar{i}} p_{\bar{i}} \partial X^j + b_{i\bar{j}} \partial X^i \bar{\partial} X^{\bar{j}})) + \int_{\Sigma} R^{(2)} \hat{\phi} \end{aligned}$$

**A.S. Losev, A.V. Marshakov, A.M.Z.**

“On first order formalism in string  
theory”

Phys.Lett.B 633(2006)375

**Complication:**  $g, b, \mu, \bar{\mu}, \hat{\phi} \longrightarrow G, B, \phi$  **nonlinear transformation:**

$$\begin{aligned}
 G_{s\bar{k}} &= g_{\bar{i}j} \bar{\mu}_s^{\bar{i}} \mu_{\bar{k}}^j + g_{s\bar{k}} - b_{s\bar{k}}, \\
 B_{s\bar{k}} &= g_{\bar{i}j} \bar{\mu}_s^{\bar{i}} \mu_{\bar{k}}^j - g_{s\bar{k}} - b_{s\bar{k}} \\
 G_{si} &= -g_{i\bar{j}} \bar{\mu}_s^{\bar{j}} - g_{s\bar{j}} \bar{\mu}_i^{\bar{j}}, \\
 G_{\bar{s}\bar{i}} &= -g_{\bar{s}j} \mu_{\bar{i}}^j - g_{\bar{i}j} \mu_{\bar{s}}^j \\
 B_{si} &= g_{s\bar{j}} \bar{\mu}_i^{\bar{j}} - g_{i\bar{j}} \bar{\mu}_s^{\bar{j}}, \\
 B_{\bar{s}\bar{i}} &= g_{\bar{i}j} \mu_{\bar{s}}^j - g_{\bar{s}j} \mu_{\bar{i}}^j, \\
 \phi &= \log \sqrt{g} + \hat{\phi}.
 \end{aligned}$$

**Advantage:**

$$S_0 = \frac{1}{2\pi h} \int_{\Sigma} d^2 z (p_i \bar{\partial} X^i + p_{\bar{i}} \partial X^{\bar{i}})$$

**VOA:**

$$X^i(z) p_j(w) \sim \frac{h \delta_j^i}{z - w}, \quad X^{\bar{i}}(\bar{z}) p_{\bar{j}}(\bar{w}) \sim \frac{h \delta_{\bar{j}}^{\bar{i}}}{\bar{z} - \bar{w}}$$

When  $\mu, \bar{\mu}, b = 0$

$$G_{i\bar{j}} = -B_{i\bar{j}} = g_{i\bar{j}}, \quad \phi = \hat{\phi} + \log \sqrt{g}$$

Resulting Einstein equations appear to be bilinear in  $g^{i\bar{j}}$

$$\begin{aligned} \partial_i \partial_{\bar{k}} \Phi_0 &= 0, & \partial_{\bar{p}} d_{\bar{l}}^{\Phi_0} g^{\bar{l}k} &= 0, & \partial_p d_l^{\Phi_0} g^{\bar{k}l} &= 0, \\ 2g^{r\bar{l}} \partial_r \partial_{\bar{l}} g^{i\bar{k}} - 2\partial_r g^{i\bar{p}} \partial_{\bar{p}} g^{r\bar{k}} - g^{i\bar{l}} \partial_{\bar{l}} d_s^{\Phi_0} g^{s\bar{k}} - \\ g^{r\bar{k}} \partial_r d_{\bar{j}}^{\Phi_0} g^{\bar{j}i} + \partial_r g^{i\bar{k}} d_{\bar{j}}^{\Phi_0} g^{\bar{j}r} + \partial_{\bar{p}} g^{\bar{k}i} d_n^{\Phi_0} g^{n\bar{p}} &= 0, \end{aligned}$$

where  $d_i^{\Phi_0} g^{i\bar{j}} \equiv \partial_i g^{i\bar{j}} - 2\partial_i \Phi_0 g^{i\bar{j}}$  and  $d_{\bar{i}}^{\Phi_0} g^{j\bar{i}} \equiv \partial_{\bar{i}} g^{j\bar{i}} - 2\partial_{\bar{i}} \Phi_0 g^{j\bar{i}}$ .

They are equivalent to:

$$Q\Phi + \frac{1}{2}\{\Phi, \Phi\} = 0, \quad [b_{-1}, [\bar{b}_{-1}, \Phi]] = g^{i\bar{j}} p_i p_{\bar{j}}$$

at the order  $h^1$ .

Symmetries (holomorphic):

$$\Phi \longrightarrow \Phi + Q\Lambda + \{\Phi, \Lambda\} + \{\Lambda, \Phi\}$$

**A.M.Z.**

“Perturbed  $\beta$ - $\gamma$  Systems  
and Complex Geometry”,  
Nucl.Phys.B 759(2006) 370

## Yang-Mills equations

$$*d_A * F = 0, \quad F = dA + A \wedge A$$

$$0 \rightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d*d} \Omega^{D-1} \xrightarrow{d} \Omega^D \rightarrow 0$$

**Yang-Mills  $C_\infty$  algebra:**

$$0 \rightarrow \mathcal{F}^0 \xrightarrow{Q_\eta} \mathcal{F}^1 \xrightarrow{Q_\eta} \mathcal{F}^2 \xrightarrow{Q_\eta} \mathcal{F}^3 \rightarrow 0$$

$$(\cdot, \cdot) : \mathcal{F}^i \otimes \mathcal{F}^j \rightarrow \mathcal{F}^{i+j}$$

$$(\cdot, \cdot, \cdot) : \mathcal{F}^i \otimes \mathcal{F}^j \otimes \mathcal{F}^k \rightarrow \mathcal{F}^{i+j+k-1}$$

$$(f_1, f_2) =$$

	$f_1$	$v$	$\mathbf{A}$	$\mathbf{V}$	$a$
$f_2$		$vw$	$\mathbf{A}w$	$\mathbf{V}w$	$aw$
$\mathbf{B}$		$v\mathbf{B}$	$(\mathbf{A}, \mathbf{B})$	$\mathbf{B} \wedge \mathbf{V}$	$\mathbf{0}$
$\mathbf{W}$		$v\mathbf{W}$	$\mathbf{A} \wedge \mathbf{W}$	$\mathbf{0}$	$\mathbf{0}$
$b$		$vb$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

$$(\mathbf{A}, \mathbf{B}) = (\mathbf{A} \wedge *d\mathbf{B}) - (\mathbf{B} \wedge *d\mathbf{A}) + d * (\mathbf{A} \wedge \mathbf{B})$$

$$(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \mathbf{A} \wedge *(\mathbf{B} \wedge \mathbf{C}) - \mathbf{C} \wedge *(\mathbf{A} \wedge \mathbf{B})$$

$$*d_A * F = 0 \iff Q_\eta \mathbf{A} + (\mathbf{A}, \mathbf{A}) + (\mathbf{A}, \mathbf{A}, \mathbf{A}) = 0$$

See e.g. **A.M.Z.**

“BV Yang-Mills as a Homotopy Chern-Simons via SFT”, Int.J.Mod.Phys.A 24 (2009) 1309;

“Conformal Field Theory and Algebraic Structure of Gauge Theory”, arXiv:0812.1840

This algebra can be obtained from

$$S = \frac{1}{2\pi h} \int_D d^2z \partial X^\mu \bar{\partial} X^\nu \eta_{\mu\nu} + \int_{\partial D} A_\mu(X) dX^\mu$$

Complex  $(\mathcal{F}^*, \mathcal{Q}_\eta)$  is a subcomplex in the BRST complex  $(C^*, \mathcal{Q}_\eta)$  of open string. LZ operations  $\mu, \nu$  reproduce operations  $(\cdot, \cdot)$  and  $(\cdot, \cdot, \cdot)$  at the level  $h^0$ .

In order to get rid of logarithms we need introduce first order formulation:

$$S = \frac{1}{2\pi h} \int_{H^+} d^2z (p_\mu \bar{\partial} X^\mu + \bar{p}_\mu \partial \bar{X}^\mu - \eta^{\mu\nu} p_\mu \bar{p}_\nu) + \int_{\mathbb{R}} dz (\mathcal{A}_\mu(X) \partial X^\mu + \mathcal{B}^\mu(X) p_\mu)$$

**Boundary conditions:**

$$p_\mu|_{\mathbb{R}} = \bar{p}_\mu|_{\mathbb{R}}, \quad X^\mu|_{\mathbb{R}} = \bar{X}^\mu|_{\mathbb{R}}$$

**VOA:**

$$X^\mu(z) p_\nu(w) \sim \frac{h \delta_\nu^\mu}{z - w}$$

**Gaussian integration:**

$$\mathcal{A}_\mu(X) \partial X^\mu + \mathcal{B}^\mu(X) p_\mu \rightarrow A_\mu(X) dX^\mu,$$

where  $A_\mu(X) = \mathcal{A}_\mu(X) + \eta_{\mu\nu} \mathcal{B}^\nu(X)$

## Extended BRST operator of open string:

$$\hat{Q}_\eta = Q_{X,p} + \eta^{\alpha\beta} \mu(a_\alpha, \{a_\beta, \cdot\}), \quad F\hat{Q}_\eta = Q_\eta F$$

where  $Q_{X,p}$  is a BRST operator for  $X$ - $p$  VOA, and  $a_\alpha = cp_\alpha$ .

It is possible to deform  $\mu$  w.r.t.  $\eta^{\alpha\beta}$  in such a way that the new operation  $\mu^\eta$  will be homotopy commutative and associative w.r.t.  $\hat{Q}_\eta$ . The complex  $(\mathcal{F}^*, Q_\eta)$  lies in the kernel of  $L_0$  (0th Virasoro mode of  $X$ - $p$  VOA):

$$\begin{aligned} \rho_u &= u(X), & \phi'_A &= cA_\mu(X)\partial X^\mu, \\ \phi''_B &= c : B^\mu(X)p_\mu :, \\ \phi_a &= \partial ca(X), & \psi'_V &= c\partial cV_\mu(X)\partial X^\mu, \\ \psi''_W &= c\partial c : W^\mu(X)p_\mu :, \\ \psi_b &= c\partial^2 cb(X), & \chi_v &= c\partial c\partial^2 cv(X). \end{aligned}$$

Combining deformed  $\mu^\eta, n^\eta$  we find that they reproduce YM  $C_\infty$  algebra on  $(\mathcal{F}^*, Q_\eta)$ .

## Relation to Courant/Dorfman algebroid:

$$\{\phi'_A + \phi''_B, \phi'_{\bar{A}} + \phi''_{\bar{B}}\} = h(\phi'_{L_{B\bar{A}} - di_{\bar{B}A}} + \phi''_{[B, \bar{B}]_{Lie}})$$

**A.M.Z.**

“ $\beta$ - $\gamma$  systems and the deformations of BRST operator”, to appear



## Chiral de Rham complex and Kodaira-Spencer Theory

VOA ( $V$ ):

$$X^i(z)p_j(w) \sim \frac{1}{z-w}, \quad \psi^i(z)\chi_j(w) \sim \frac{1}{z-w}$$

Chiral de Rham cohomology operator:

$$Q = \frac{1}{2\pi i} \oint \psi^i p_i$$

Bilinear operation  $\{a, b\} = \frac{(-1)^{|a|}}{2\pi i} \oint dz [Q, a](z)b$   
satisfies Gerstenhaber algebra together with  $\mu$ .  $\{\cdot, \cdot\}$  reproduces Schouten bracket on the operators  $f^{i_1 \dots i_n}(x)\chi_{i_1} \dots \chi_{i_n}$

**F. Malikov**

“Lagrangian approach to Sheaves on  
Vertex Algebras”

Comm.Math.Phys. 278(2008) 487

Introducing antichiral part ( $\bar{V}$ ):

$$X^{\bar{i}}(\bar{z})p_{\bar{j}}(\bar{w}) \sim \frac{1}{\bar{z}-\bar{w}}, \quad \psi^{\bar{i}}(\bar{z})\chi_{\bar{j}}(\bar{w}) \sim \frac{1}{\bar{z}-\bar{w}}$$

Let us consider elements of the form

$$\xi_{\bar{j}_1 \dots \bar{j}_n}^{i_1 \dots i_n}(X, \bar{X})\chi_{i_1} \dots \chi_{i_n} \psi^{\bar{j}_1} \dots \psi^{\bar{j}_n} \in V \otimes \bar{V}$$

Then

$\bar{Q}\mu + \{\mu, \mu\} = 0$ ,  $\mu = \mu_j^i(X, \bar{X})\chi_i\psi^{\bar{j}}$  coincides with  
Kodaira-Spencer equation.

Actually, most of Barannikov-Kontsevich formulas

**S. Barannikov, M. Kontsevich**

“Frobenius Manifolds and Formality of  
Lie algebras of Polyvector Fields”,

alg-geom/9710032

may be reproduced on VOA language.

## Conclusions

- Physical interpretation of “higher homotopies” in Lian-Zuckerman construction.
- Construction of the field theory equations (Einstein, YM, Kodaira-Spencer) via the deformation theory of semi-infinite cohomology operator for certain VOAs.
- Algebraic approach to the study of  $\beta$ -function in  $\sigma$ -models.
- Relation of Courant/Dorfman algebroid and Yang-Mills  $C_\infty$ -algebra.