A Counterexample to the Transfer Conjecture

by

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In this paper we discuss the following conjecture, which has been attributed to D. Quillen, and present a counterexample to it.

<u>Conjecture</u>: If a representable homotopy functor admits a transfer, then it extends to a cohomology theory.

This conjecture appeared in a preliminary version (circa 1970) of [S2], at that time entitled "Homotopy everything H spaces." In that preprint, G. Segal presented his permutative category approach to the study of infinite loop spaces. As an application of his methods, he outlined a supposed proof of this conjecture. However, as work on transfer and infinite loop spaces continued in the early 1970's, many doubts were raised about the truth of this conjecture.

If $X = \Omega Y$ then X has an associative H space structure and the functor $h(\ ) = [\ ,X]$ takes values in the category of groups. Conversely if $h(\ ) = [\ ,X]$ takes values in the category of groups, then X has a homotopy associative H structure, but is not necessarily of the homotopy type of a loop space. Stasheff [S4] introduced the notion of A_k structures and showed that a connected space X is homotopy associative if and only if it has an A_3 structure, while X has the homotopy type of a loop space if and only if it has an A_∞ structure.

Assume that there is a sequence $\{X_n\}$ with $\Omega X_n = X_{n-1}$. Then we say that X_0 is a perfect infinite loop space. In this case X has a Q algebra structure (Definition 1.1 and Theorem 1.2), and $h(\ ) = [\ ,X]$ admits a transfer. Conversely if $h(\ ) = [\ ,X]$ admits a transfer, then we say X is a transfer space. In sections 2 and 3 we introduce the notion of Q_k structures. A connected space X is a transfer space if and only if it has a Q_2 structure while X is equivalent to an infinite loop space if and only if X has a Q_∞ structure (Theorems 2.4 and 2.5).

If $X = \Omega Y$ then the Eilenberg-Moore spectral sequence arising from the Milnor construction converges to $H^*(Y;\Lambda)$ from a functor of the coalgebra $H^*(X;\Lambda)$. Stasheff [S4] found a relationship between k cycles and A_{k+1} structures and was able to use this to construct an A_k space with no A_{k+1} structure.

If X is a perfect infinite loop space, then Haynes Miller has constructed an infinite delooping spectral sequence [M8]. We prove that k cycles in this spectral sequence correspond to Q_k maps and that these induce Q_k spaces (Theorems 3.3 and 5.2). Thus to construct a counterexample to the transfer conjecture, we need to find a 2 cycle which is not a 3 cycle in the Miller spectral sequence.

In an earlier draft we were able to construct a 2 stage Postnikov system P and a map $f: P \to K(Z/p,n)$ which represented a 2 cycle, but not a 3 cycle. The induced fiber space E turns out to be a 3 stage Postnikov system such that $h(\ ) = [\ ,E]$ is a counterexample to the transfer conjecture. The proof required a great deal of ad hoc technical constructions and proofs.

Recently work of Madsen and Snaith ([MST],[M4]) was brought to our attention. They show that if s_k is the primitive generator of $H^{2k}(BU;Z_{(2)})$, then $2s_k$ is represented by a transfer commuting map, which we also write $2s_k: BU \to K(Z_{(2)},2k)$. Furthermore, they showed that the fiber of $2s_7$ was not an infinite loop space. Unfortunately, this fiber was not a transfer space so no counterexample was obtained.

Our techniques apply directly to this type of construction. We show that the fiber induced by $4s_k$ is a transfer space for all k, but that the fiber E of $4s_{15}$ does not have an infinite loop structure.

Thus, [,E] does give a counterexample to the transfer conjecture. The proof of this result is far less technical than our proof for the Postnikov system counterexample.

In section 1 we review infinite loop space theory and Q algebras. If (X,ρ) is a Q algebra, then following Beck [B] and May [M6] we construct a simplicial spectrum $B(\Sigma^{\infty},Q,X)_*$ whose realization gives an infinite delooping of X. In sections 2 and 3 we introduce Q_k structures and establish its relationship with transfer. In sections 4 and 5 the spectral sequence of the simplicial spectrum of section 1 is constructed and the relationship between its cycles and Q_k maps is proven. In sections 6 and 7 we prove that under suitable conditions $E_r^{s,t}(X,Z_{(p)})$ is a Z/p module for $s \geq 1$ and $r \geq 2$. This fact allows us to conclude that $4s_k$ is a Q_2 map and so its induced fiber E is a transfer space. In section 8 we prove that although E has a 2 fold loop sturcture it does not have a 3 fold loop structure. This implies that h( ) = [,E] is a counterexample to the transfer conjecture.

In our constructions we rely heavily on the theory of infinite loop spaces built up by P. May. The volumes [M6] and [M7] serve as background references for much of this paper.

We appreciate the interest and encouragement of J. Stasheff and P. May. We are grateful to I. Madsen for his discovery of an error in a previous manuscript. It was in the understanding and correction of that error that we were led to our present spectral sequence approach. We feel that the Miller delooping spectral sequence will have many interesting applications.

§1. Infinite loop spaces and Beck's Theorem

We will work throughout in the category of pointed compactly generated spaces with $H_*(X)$ of finite type. By a representable homotopy functor h we mean $h(Y) = [Y,X]$ = based homotopy classes of maps where X is determined up to weak homotopy type. We say that h extends to a cohomology theory if there is a sequence $\{h^k\}$ with $h^k(\Sigma Y)$ naturally equivalent to $h^{k-1}(Y)$ and h^0 naturally equivalent to h.

By an Ω spectrum we will mean a sequence of spaces $\{X_k\}$ which are connected for $k > 0$ and such that there are weak homotopy equivalences $\Omega X_k \simeq X_{k-1}$ for $k > 0$. If $\{X_k\}$ is an Ω spectrum, let $\Omega^\infty\{X_k\} = X_0$ be the 0th space. If $X \simeq \Omega^\infty\{X_k\}$ for some Ω spectrum then we say that X is an infinite loop space. The following classical result follows immediately using the adjointness of Σ and Ω.

Proposition 1.1. The functor $h(Y) = [Y,X]$ extends to a cohomology theory if and only if X is an infinite loop space.

We say that the Ω spectrum $\{X_k\}$ is perfect if $\Omega X_k = X_{k-1}$ for $k > 0$. For any space Y let $\Sigma^\infty Y$ be the perfect Ω spectrum $\{Q\Sigma^k Y\}$ where $Q = \varinjlim \Omega^N\Sigma^N$. The adjunctions between Σ and Ω induce the adjunctions

$$\eta\colon 1 \to \Omega^\infty\Sigma^\infty = Q$$

$$\text{and} \qquad \varepsilon\colon \Sigma^\infty\Omega^\infty \to 1 \quad \text{(see [M5])}.$$

Let $\mu = \Omega^\infty\varepsilon\Sigma^\infty\colon Q^2 \to Q$. Then for functorial reasons there are identities $\mu Q\eta = 1 = \mu\eta Q$ and $\mu\mu Q = \mu Q\mu$. We call (Q,μ,η) a monad.

Definition 1.2. If there is a map $\rho\colon QX \to X$ satisfying $\rho\eta = 1_X$ and $\rho\mu = \rho Q\rho\colon Q^2X \to X$ then we say that (X,ρ) is a Q algebra and that X has a Q structure.

A more complete description of these concepts and of the important application below can be found in [B] and §2 [M6]. See also the more general treatment in Chapter VI [M2].

Theorem 1.3. (Beck). If X is a perfect infinite loop space, then X has a Q structure. If (X,ρ) is a connected Q algebra, then X is an infinite loop space.

Proof. Assume $\{X_k\}$ satisfies $\Omega X_k = X_{k-1}$ and $X_0 = X$. Define $\rho: QX \to X$ to be the limit of $\rho_N: \Omega^N \Sigma^N X_0 = \Omega^N \Sigma^N \Omega^N X_N \xrightarrow{\Omega^N \varepsilon_N} \Omega^N X_N = X_0$. The verification that (X,ρ) is a Q algebra is standard [M6].

Conversely assume that (X,ρ) is a Q algebra. Define the simplicial Ω spectrum $B_* = B(\Sigma^\infty, Q, X)_*$ by

$$B_q = \Sigma^\infty Q^q X \quad \text{and}$$

$$(1.4) \qquad \partial_i = \begin{cases} \varepsilon\Sigma^\infty Q^{q-1} & \text{if } i = 0 \\ \Sigma^\infty Q^{i-1}\mu Q^j & \text{if } 0 < i < q \text{ and } i + j = q - 1 \\ \Sigma^\infty Q^{q-1}\rho & \text{if } i = q \end{cases}$$

$$s_i = \Sigma^\infty Q^i \eta Q^j \text{ for } i + j = q.$$

The realization $||B_*||$ is the Ω spectrum $\{X_k\}$ with X_k defined by

$$\coprod_{q \geq 0} \Delta^q \times Q\,\Sigma^k Q^q X/\sim$$

where $\sim$ is the standard equivalence relation $(u,\partial_i x) \sim (\delta_i u, x)$ and $(u,s_i x) \sim (\sigma_i u, x)$. By §12 [M6], this is indeed an Ω spectrum.

Furthermore if X is connected then the inclusion of $X = \Delta^\circ \times Q^\circ X$ into $\coprod_q \Delta^q \times Q^{q+1}X/\sim = X_o = \Omega^\infty ||B*||$ is a strong deformation retraction by Theorem 9.10 [M6].

The fact that many infinite loop spaces do not have a (strict) Q algebra structure has necessitated the introduction of various infinite loop space machines, such as those by Boardman and Vogt, May and Segal. Further generalizations by the second author are discussed in the next section. On the other hand, if $\{X_n\}$ is an Ω spectrum, then $\{\varinjlim \Omega^N X_{n+N}\}$ is a perfect Ω spectrum equivalent to $\{X_n\}$ up to weak homotopy [M5]. Thus we may replace infinite loop spaces by Q algebras which contain the same homotopy theoretical information.

§2. Transfer and Q_k Spaces

We now consider the definition of transfer for a representable homotopy functor and discuss its relationship with infinite loop space structures. See also [L1] and [R].

Definition 2.1. We say that the functor $h(\_) = [\_,X]$ admits a transfer for finite coverings if given a covering $p: \tilde{Y} \to Y$, there is a map of pointed sets $\tau_p: [\tilde{Y},X] \to [Y,X]$ such that

1) τ is natural with respect to pullbacks
2) If $id: Y \to Y$ is the identity covering, then $\tau_{id} = id$
3) Given a composition of coverings $\tilde{\tilde{Y}} \xrightarrow{p_2} \tilde{Y} \xrightarrow{p_1} Y$, then $\tau_{p_1 \circ p_2} = \tau_{p_1} \circ \tau_{p_2}$
4) Given the covering $p = id \coprod id: X \coprod X \to X$, then $\tau_p[id \coprod *] = [id] = \tau_p[* \coprod id]$, where $\coprod$ means disjoint union and $*$ denotes the constant map $X \to *$, the basepoint of X.

With this definition, one can immediately deduce

Proposition 2.2. If the functor $h(\_) = [\_,X]$ admits a transfer, then h takes on values in the category of abelian monoids. [E, pp. 12-13], [L1 , pp. 54-62], [M4].

Remark: This proposition implies that X is a homotopy associative, homotopy commutative H space.

A generalization up to homotopy of Q structures on spaces has been developed by Lada in [L2] and may be summarized by the following definition and theorem.

Definition 2.3. A space X is a Q_k space if there is a family of homotopies

$$h_q: I^q \times Q^{q+1}X \to X \quad \text{for } q < k$$

such that

$$h_q(t_1,\ldots,t_q,z) = h_{q-1} \circ (1 \times Q^{j-1}\mu Q^{q-j})(t_1,\ldots,\hat{t}_j,\ldots,t_q,z) \text{ if } t_j=0,$$

and $h_q(t_1,\ldots,t_q,z) = h_{j-1} \circ (1 \times Q^j h_{q-j})(t_1,\ldots,\hat{t}_j,\ldots,t_q,z)$ if $t_j=1$,

and $h_o \circ \eta = id: X \to X$.

Note that h_o: $QX \to X$ is a retraction and that the homotopy $h_1: I \times Q^2X \to X$ requires only that $\rho\mu$ be homotopic to $\rho Q\rho$ where $h_o = \rho$.

Theorem 2.4. A connected space X is an infinite loop space if and only if X has a Q_∞ structure, i.e., a Q_k structure for all k.

With the above definitions in hand, we are now able to discuss the relationship between transfer and infinite loop space structures. The following theorem has been proven by a number of authors, [E], [KP], [L1], [M4], [R].

Theorem 2.5. The functor $h(\_) = [\_,X]$ admits a transfer if and only if X is a Q_2 space.

For this reason Madsen calls a Q_2 space a transfer space. Thus the transfer conjecture can be reformulated as follows.

Conjecture′ Every Q_2 structure on X extends to a Q_∞ structure.

It is when the conjecture is stated in this form that it appears unlikely to be true. To find a counterexample, all that one needs is a space X that is not an infinite loop space and yet possesses a Q_2 structure. The remainder of this section is occupied with a sketch of the main ideas in the proof of Theorem 2.5.

Let $W\Sigma_n$ be the normalized Milnor construction for Σ_n, the symmetric group on n symbols. We may regard QX as $\coprod(W\Sigma_n \times X^n)/\sim$ by the results of the preprint version of [DL] and [M7, §4]. If [_,X] admits a transfer, a Q_2 structure for X may be defined by the following argument; see [L1] for details. Consider the n-fold covering $p_n: W\Sigma_n \times X^n \times F_n \longrightarrow W\Sigma_n \times X^n$ where $F_n = \{1,\dots,n\}$, and $p_n = id$ on each part of the union. Define maps $f_n: W\Sigma_n \times X^n \times F_n \to X$ by projection of a tuple indexed by $i \in F_n$ onto the i^{th} coordinate in X^n. One may then carefully choose equivalent elements of $\tau_{p_n}[f_n]$ to serve as building blocks of $h_o: QX \to X$.

To construct the homotopy $h_1: I \times Q^2X \to X$, consider the composition

$$W\Sigma_k \times W\Sigma_{j_1} \times X^{j_1} \times \dots \times W\Sigma_{j_k} \times X^{j_k} \times F_j \xrightarrow{\coprod_{i=1}^{k} 1 \times p_{j_i}}$$

$$W\Sigma_k \times W\Sigma_{j_1} \times X^{j_1} \times \dots \times W\Sigma_{j_k} \times X^{j_k} \times F_k \xrightarrow{p_k}$$

$$W\Sigma_k \times W\Sigma_{j_1} \times X^{j_1} \times \dots \times W\Sigma_{j_k} \times X^{j_k} \quad \text{where } j = \sum_{i=1}^{k} j_i.$$ One then computes transfer of f_j through the composition and compares the answer with the result from applying the naturality property of transfer to the pullback diagram

$$\begin{array}{ccc} W\Sigma_k \times W\Sigma_{j_1} \times \dots \times W\Sigma_{j_k} \times X^j \times F_j & \xrightarrow{\overline{\gamma}} & W\Sigma_j \times X^j \times F_j \\ \downarrow & & \downarrow \\ W\Sigma_k \times W\Sigma_{j_1} \times \dots \times W\Sigma_{j_k} \times X^j & \xrightarrow{\gamma} & W\Sigma_j \times X^j \end{array} ;$$

here γ is induced by a generalized wreath product. The composition property of transfer will then yield the requisite homotopy.

To see that a Q_2 structure implies the existence of a transfer map, we sketch Kahn and Priddy's approach. Let $p: \tilde{Y} \to Y$ be an n-fold covering and $P(\tilde{Y})$ the associated principal Σ_n covering. One then composes the obvious map $Y \to P(\tilde{Y}) \times_{\Sigma_n} \tilde{Y}^n$ with the classifying map $P(\tilde{Y}) \to W\Sigma_n$ to obtain a map $\bar{p}: Y \to W\Sigma_n \times_{\Sigma_n} \tilde{Y}^n$. If $f: \tilde{Y} \to X$ is a map, the transfer of f may be represented by the composition

$$Y \longrightarrow W\Sigma_n \times_{\Sigma_n} \tilde{Y}^n \xrightarrow{1 \times f^n} W\Sigma_n \times_{\Sigma_n} X^n \xrightarrow{h_o} X$$

where h_o is the Dyer Lashof map induced by $h_o: QX \to X$ [DL]. Properties 1, 2 and 4 of transfer may be readily deduced from this construction. To verify the composition property, property 3, let $q: \tilde{\tilde{Y}} \to Y$ be an m-fold covering. One then shows that $\overline{p \circ q}$ is the compostion

$$Y \xrightarrow{\bar{p}} W\Sigma_n \times_{\Sigma_n} \tilde{Y}^n \xrightarrow{1 \times \bar{q}^n} W\Sigma_n \times_{\Sigma_n} (W\Sigma_m \times_{\Sigma_m} \tilde{\tilde{Y}}^m)^n \quad \text{and that}$$

$$W\Sigma_n \times_{\Sigma_n} (W\Sigma_m \times_{\Sigma_m} \tilde{\tilde{Y}}^m)^n = W(\Sigma_n \int \Sigma_m) \times_{\Sigma_n \int \Sigma_m} \tilde{\tilde{Y}}^{nm}$$

where $\Sigma_n \int \Sigma_m \subset \Sigma_{mn}$ is the wreath product.

§3. Q_k maps induce Q_k spaces

Assume that (X,ρ) and (K,ϕ) are Q algebras. If $f:X \to K$ commutes with the algebra structure, i.e. if $\phi Qf = f\rho : QX \to K$, then it is easy to see that f extends to a map of simplicial spectra $f_*:B(\Sigma^\infty,Q,X)_* \to B(\Sigma^\infty,Q,K)_*$, and thus to their realizations. In this case f is a stable, or infinite loop map. Also in this case it is easy to verify that the fiber in $E \xrightarrow{\pi} X \xrightarrow{f} K$ is a Q algebra.

Definition 3.1. A map $f:(X,\rho) \to (K,\phi)$ between Q algebras is called a Q_k map if there is a collection of maps

$$f_q:\Delta^q \times Q^qX \to K \quad \text{for } q \le k$$

with $f_o = f$ and satisfying

$$f_q(t_o,\ldots,t_q,z) = \begin{cases} \phi Qf_{q-1}(t_1,\ldots,t_q,z) & \text{if } t_o=0 \\ f_{q-1}(t_o,\ldots,\hat{t}_i,\ldots,t_q,Q^{i-1}\mu Q^j z) & \text{if } t_i=0,\ i+j=q- \\ f_{q-1}(t_o,\ldots,t_{q-1},Q^{q-1}\rho z) & \text{if } t_q=0 \end{cases}$$

where $Q^{i-1}\mu Q^j:Q^{i-1}Q^2Q^jX \to Q^{i-1}QQ^jX$.

If f is a Q_k map for all k, then we say that f is a strong homotopy Q (or shQ) map. Such maps can be lifted to infinite loop maps [L2]. If f is a Q_1 map, then we only require that $\phi Qf \simeq f\rho$. Such maps have been called transfer commuting maps [MST], [M4]. We will be most interested in Q_2 maps in this paper.

Let $\underline{F_k} = \coprod_{q \le k} \Delta^q \times \Sigma^\infty Q^qX/\sim$ be a filtration of $||B_*||$. Let $\underline{K}$ be an Ω spectrum and let (K,ϕ) be the Q algebra constructed in Theorem 1.3. By a map $G:\underline{F_k} \to \underline{K}$ we mean a sequence of compositions for $q \le k$ $\Delta^q \times \Sigma^\infty Q^qX \xrightarrow{i} \Sigma^\infty(\Delta^q \times Q^qX) \xrightarrow{g_q} \underline{K}$ where g_q is a map of spectra satisfying the coherence relations induced by (1.4).

The adjoint of $g_q : \Sigma^{\infty}(\Delta^q \times Q^q X) \longrightarrow \underline{K}$ is the map f_q given by the composition $\Delta^q \times Q^q X \xrightarrow{\eta} \Omega^{\infty}\Sigma^{\infty}(\Delta^q \times Q^q X) \xrightarrow{\Omega^{\infty} g_q} \Omega^{\infty}\underline{K} = K$. The compatibility conditions of (1.4) for g_q translate to the compatibility relations (3.1) which imply that f is a Q_k map. Conversely if $f:(X,\rho) \longrightarrow (K,\phi)$ is a Q_k map, then we may construct a map $G:\underline{F_k} \to \underline{K}$ by defining g_q to be the composition

$$\Sigma^{\infty}(\Delta^q \times Q^q X) \xrightarrow{\Sigma^{\infty} f_q} \Sigma^{\infty} K = \Sigma^{\infty}\Omega^{\infty}\underline{K} \xrightarrow{\varepsilon} K.$$

Compare Chapter IV [M2]. Thus we have proven the following.

Theorem 3.2. A map $f:(X,\rho) \longrightarrow (K,\phi)$ is a Q_k map if and only if there is a map $G:\underline{F_k} \longrightarrow \underline{K}$ such that f is the adjoint of $g_o : \Sigma^{\infty} X \longrightarrow \underline{K}$.

The Q_k spaces and maps are, of course, analogues of Stasheff's higher homotopy associative (A_k) spaces and maps. We now prove the analogue of his Theorem 6.1 [S4].

Theorem 3.3. Let $f:(X,\rho) \to (K,\phi)$ be a Q_k map between connected Q algebras. Let E be the fiber space over X induced by f from the path loop fibration over K. Then E is a Q_k space.

Proof: The Q_k structure on a space E involves maps $I^q \times Q^{q+1}E \to E$. It will be helpful in the proof if we translate the definition of Q_k maps from the simplicial to the cubical theory. Indeed one can check that f is a Q_k map if there are maps $f_q : I^q \times Q^q X \to K$ such that $f_o = f$ and

$$f_q(t_1,\ldots,t_q,z) = \begin{cases} \phi Q f_{q-1}(t_2,\ldots,t_q,z) & \text{if } t_1=0 \\ f_{q-1}(t_1,\ldots,\hat{t}_i,\ldots,t_q,Q^{i-1}\mu Q^j z) & \text{if } t_i=0,\ i+j=q \\ f_{i-1}(t_1,\ldots,t_{i-1},Q^{i-1}\rho^j z) & \text{if } t_i=1,\ i+j=q \end{cases}$$

where $\rho^i = \rho Q\rho \ldots Q^i\rho$.

We present the details of the proof of this theorem only for the case when $k=2$ as that is all that is required in subsequent sections. The existence of a Q_2 structure on f implies that we have homotopies

$$f_1 : I \times QX \to K \quad \text{and} \quad f_2 : I^2 \times Q^2X \to K$$

such that $f_1(0,z) = \phi Qf(z)$, $f_1(1,z) = f\rho(z)$, $f_2(0,t,z) = \phi Qf_1(t,z)$, $f_2(t,0,z) = f_1(t,\mu z)$, $f_2(1,t,z) = f\rho Q\rho(z)$, $f_2(t,1,z) = f_1(t,Q\rho z)$.

We regard $E \subset X \times PK$ to be defined by $\{(x,\lambda)\,|\,f(x) = \lambda(1)\}$. Note that by [M6, p. 6] we have $\rho_{PK}:QPK \to PK$ defined by $\rho_{PK}(z,\lambda_1,\ldots,\lambda_n)(s) = \rho_K(z,\lambda_1(s),\ldots,\lambda_n(s))$ where $(z,\lambda_1,\ldots,\lambda_n) \in W\Sigma_n \times (PK)^n$.

The Q structure map for E, $\rho_E:QE \to E$, may now be defined by

$$\rho_E(z,x_1,\lambda_1,\ldots,x_n,\lambda_n) =$$

$$(\rho(z,x_1,\ldots,x_n),\ \rho_{PK}(z,\lambda_1,\ldots,\lambda_n)(\cdot) + f_1(\cdot,z,x_1,\ldots,x_n)).$$

To see that the path addition in the definition is well defined, note that $\rho_{PK}(z,\lambda_1,\ldots,\lambda_n)(1) = \phi(z,\lambda_1(1),\ldots,\lambda_n(1)) = \phi(z,f(x_1),\ldots,f(x_n)) = \phi Qf(z,x_1,\ldots,x_n) = f_1(0,z,x_1,\ldots,x_n)$. The remaining step is to verify that $\rho_E(z,x_1,\lambda_1,\ldots,x_n,\lambda_n) \in E$. We have $[\rho_{PK}(z,\lambda_1,\ldots,\lambda_n) + f_1(z,x_1,\ldots,x_n)](1) = f_1(1,z,x_1,\ldots,x_n) = f \circ \rho(z,x_1,\ldots,x_n)$ and we are done.

At this point we have shown that a Q_1 structure on f yields a Q_1 structure on E. It will be useful to observe that the second coordinate of ρ_E may be described by

$$g(t) = \begin{cases} \rho_{PK}(2t) & t \leq 1/2 \\ f_1(2t-1) & t \geq 1/2 \end{cases}$$

We now turn our attention to the Q_2 structure on E. A point in Q^2E lies in some subspace of the form $W\Sigma_k \times (\coprod_n W\Sigma_n \times E^n/\sim)^k$. Denote such a point by $(w,z_1,\ldots,z_k,\overline{x}_{\alpha_1},\overline{\lambda}_{\alpha_1},\ldots,\overline{x}_{\alpha_k},\overline{\lambda}_{\alpha_k})$ where $\overline{x}_{\alpha_j}$ and $\overline{\lambda}_{\alpha_j}$ are i_j tuples indexed by the integers $\sum_{n=1}^{j-1} i_n + \ell$, $\ell=1,\ldots,i_j$ and the p^{th} component of each $\overline{x}_{\alpha_j}$ and $\overline{\lambda}_{\alpha_j}$ satisfy $f(\overline{x}_{\alpha_j})_p = (\overline{\lambda}_{\alpha_j})_p(1)$. We also have $z_j \in W\Sigma_{i_j}$. Let us now define $\rho_E^1 : I \times Q^2E \to E$ by

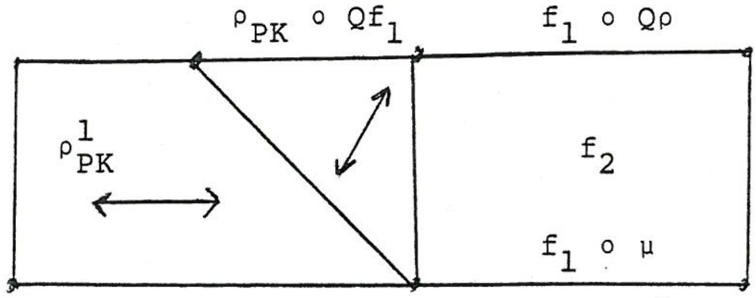
$$\rho_E^1(t_2,w,z_1,\ldots,z_k,\overline{x}_{\alpha_1},\overline{\lambda}_{\alpha_1},\ldots,\overline{x}_{\alpha_k},\overline{\lambda}_{\alpha_k})(t_1) = (\rho_X^1(w,z_1,\ldots,z_k,$$

$$\overline{x}_{\alpha_1},\ldots,\overline{x}_{\alpha_k}),\ g_1(t_1,t_2,w,z_1,\ldots,z_k,\overline{x}_{\alpha_1},\overline{\lambda}_{\alpha_1},\ldots,\overline{x}_{\alpha_k},\overline{\lambda}_{\alpha_k}))$$

where

$$g_1(t_1,t_2) = \begin{cases} \rho_{PK}^1\left(\dfrac{4t_1}{2-t_2}\right) & 0 \leq t_1 \leq \dfrac{2-t_2}{4} \\ \rho_{PK} \circ Qf_1(4t_1 - (2-t_2)) & \dfrac{2-t_2}{4} \leq t_1 \leq \dfrac{1}{2} \\ f_2(2t_1-1,t_2) & t_1 \geq \dfrac{1}{2} \end{cases}$$

We regard the first coordinate, t_1, as the path parameter. The following diagram should clarify the construction of g_1.



Note that it is easy to see that the image of ρ^1_E lies in E. To see that ρ^1_E does indeed define a Q_2 structure for E_1 we have to verify the compatibility requirements set out in Definition 2.3. We need only check the second coordinate of ρ^1_E, g_1, as compatibility is obviously satisfied in the first.

Suppose $t_2 = 0$. Then

$$g_1(t_1,0) = \begin{cases} \rho^1_{PK}(2t_1) & 0 \leq t_1 \leq \frac{1}{2} \\ \rho_{PK} \circ Qf_1(4t_1-1) & t_1 = \frac{1}{2} \\ f_2(2t_1-1,0) & t_1 \geq \frac{1}{2} \end{cases}$$

$$= \begin{cases} \rho_{PK} \circ \mu(2t_1) & 0 \leq t_1 \leq \frac{1}{2} \\ f_1(2t_1-1) \circ \mu & \frac{1}{2} \leq t_1 \leq 1 \end{cases}$$

$= g \circ \mu$. Thus $\rho^1_E|_0 = \rho_E \circ \mu$.

Now let $t_2 = 1$. Then

$$g_1(t_1,1) = \begin{cases} \rho^1_{PK}(4t_1) & 0 \leq t_1 \leq \frac{1}{4} \\ \rho_{PK} \circ Qf_1(4t_1-1) & \frac{1}{4} \leq t_1 \leq \frac{1}{2} \\ f_2(2t_1-1,1) & t_1 \geq \frac{1}{2} \end{cases}$$

$= (\rho^1_{PK} + \rho_{PK} \circ Qf_1) + f_1 \circ Q\rho_X$. A straightforward calculation shows that this path is indeed the second coordinate of $\rho_E \circ Q\rho_E$.

Remark: In general assume that $f_q : I^q \times Q^q X \to K$, $q \leq k$ is a Q_k map. It is possible to define a Q_k structure on E, $\rho_E^q : I^q \times Q^{q+1} E \to E$ by $\rho_E^q = (\rho_X^q, g_q)$ where

$$g_q(t_1,\ldots,t_{q+1}) = \begin{cases} \rho_{PK}^{j-1} \circ Q^j f_{q-j+1}\left(\dfrac{2^{j+1} t_1 - \prod_{i=2}^{j+1} (2-t_i)}{\prod_{i=2}^{j} (2-t_i)}, t_{j+2},\ldots,t_{q+1} \right) \\ \qquad \text{if } \dfrac{\prod_{i=2}^{j+1} (2-t_i)}{2^{j+1}} \leq t_1 \leq \dfrac{\prod_{i=2}^{j} (2-t_i)}{2^j} \\ \rho_{PK}^q \left(\dfrac{2^{q+1} t_1}{\prod_{i=2}^{q+1} (2-t_i)} \right) \qquad \text{if } t_1 \leq \dfrac{\prod_{i=2}^{q+1} (2-t_i)}{2^{q+1}} \end{cases}$$

We let $j=0,1,\ldots,q$ and define $\prod_{i=2}^{1} (2-t_i) = 1 = \prod_{i=2}^{0} (2-t_i)$. The verification of the coherence formulas is quite tedious and in the interests of good taste will be omitted.

§4. Stable homology and the delooping spectral sequence

Let h_* be a connected homology theory, i.e., $h_q(X) = 0$ for $q \leq 0$. If $\underline{X} = \{X_k\}$ is an Ω spectrum, then we define the stable homology by

$$h_q^S(\underline{X}) = \lim_{\rightarrow} h_{q+k}(X_k)$$
$$= h_{q+k}(X_k) \quad \text{for } k > q.$$

If (X,ρ) is a Q algebra, then we define $h_q^S(X,\rho) = h_q^S(||B_*||)$.

Proposition 4.1. For a connected space X there is a suspension isomorphism

$$\Sigma: h_q(X) \approx h_q{}^S(\Sigma^\infty X).$$

Proof: Note that $\eta: \Sigma^n X \to Q\Sigma^n X$ is a 2n-1 equivalence. Thus

$$h_q^S(\Sigma^\infty X) = h_{q+n}(Q\Sigma^n X) \quad \text{for } n > q$$
$$= h_{q+n}(\Sigma^n X) = h_q(X).$$

In an analogous way we may define the stable cohomology functor h_S^*. The usual duality and universal coefficient theorems hold.

Example 4.2. Let $h_q(\ ) = H_q(\ ;Z/p)$. If X is connected, then $H_q(QX; Z/p) = A\ T\ H_*(X;Z/p)$, the free commutative algebra on the free admissible Dyer Lashof module on $\hat{H}_*(X,Z/p)$ (see p. 42. [M7] and [DL]).

$$H_*^S(QX;Z/p) = H_*^S(\Sigma^\infty X;Z/p)$$
$$= \tilde{H}_*(X;Z/p).$$

Example 4.3. Let $\underline{K}(Z/p)$ be the perfect Ω spectrum $\{K(Z/p,n)\}$. Then $H_S^*(\underline{K}(Z/p),Z/p) = \lim H^{*+n}(K(Z/p,n),Z/p) \simeq A(p)$, the mod p Steenrod algebra.

On the other hand $H^q(K(Z/p,0);Z/p) = Z/p[Z/p]$ if $q=0$ and 0 otherwise.

Example 4.4. Let $bu = \{BU[2n,\ldots,\infty]\}$ be the Ω spectrum for connected K theory. Then $H^*(BU;Z/2) = Z/2[c_1,c_2,\ldots]$ where c_k is the Chern class of degree 2k. Adams [A1] [AP] has computed that $H_S^*(bu;Z/2) \simeq \Sigma^2 A/A(Sq^1,Sq^3)$. Results on the localized unstable and stable theories $[BU_{(p)},BU_{(p)}]$ and $[bu_{(p)},bu_{(p)}]$ have been obtained in [MST].

Let Y_* be a simplicial space and let $\partial \subset Y_q$ generically denote the subspace of degeneracies $\partial = \bigcup_i \mathrm{Im}\, s_i(Y_{q-1})$. We will assume that the inclusion is a cofibration. For a homology theory h_*, there is an associated spectral sequence with $E^1_{s,t} \simeq h_s(Y_t,\partial)$ which converges to $h_*(||Y_*||)$ [S1]. This construction can be generalized to simplicial spectra.

Theorem 4.5. Let (X,ρ) be a Q algebra and h_* a connective homology theory. Then there is a first quadrant spectral sequence with $E^1_{s,t} = E^1_{s,t}(X,h_*) \simeq h_t(Q^sX,\partial)$ which converges to $h_*^S(||B_*||) = h^S(X,\rho)$. Moreover the differential d^1 is induced by $\sum_{i=1}^{s} (-1)^i \zeta_{i*}$ where $\zeta: Q^sX \to Q^{s-1}X$ is given by

$$\zeta_i = \begin{cases} Q^{i-1}\mu Q^j & \text{for } 1 \leq i < s,\ i+j = s-1 \\ Q^{s-1}\rho & \text{for } i=s. \end{cases}$$

Proof: Recall from section 3 that $\underline{F}_s = \coprod_{q \leq s} \Delta^q \times \Sigma^\infty Q^q X/\sim$ is a filtration of $||B_*||$. Thus there is an exact couple

$$(4.6)\qquad \begin{array}{ccc} h^S_*(\underline{F}_{s-1}) & \xrightarrow{\ i_*\ } & h^S_*(\underline{F}_s) \\ & \nwarrow \quad \swarrow & \\ & h^S_*(\underline{F}_s,\underline{F}_{s-1}) & \end{array}$$

with an associated spectral sequence having $E^1_{s,t} = h^S_{s+t}(\underline{F}_s,\underline{F}_{s-1})$ and converging to $h^S_*(||B_*||)$. Moreover

$$h^S_{s+t}(\underline{F}_s,\underline{F}_{s-1}) = \lim h_{s+t+n}(\coprod_{q \leq s} \Delta^q \times Q\Sigma^n Q^q X,\_)$$

$$= h_{s+t}(\coprod_{q \leq s} \Delta^q \times Q^q X,\_)$$

$$= h_t(Q^s X,\partial).$$

(compare [S1] and [M1]).

The differential d^1: $h^S_{s+t}(\underline{F}_s,\underline{F}_{s-1}) \to h^S_{s+t}(\underline{F}_{s-1},\underline{F}_{s-2})$ is induced from the alternating sum of the maps ∂_i: $\Sigma^\infty Q^s X \to \Sigma^\infty Q^{s-1}X$ defined in (1.4). If $i > 0$ then $\partial_i = \Sigma^\infty \zeta_i$. If $i=0$ then $\partial_0 = \varepsilon\Sigma^\infty Q^{s-1}$: $\Sigma^\infty Q^s X \to \Sigma^\infty Q^{s-1}X$ can be seen to induce the 0 map on $h^S_*(\Sigma^\infty Q^s X)/(\eta Q^{s-1})_* h^S_*(\Sigma^\infty Q^{s-1}X)$, and thus on $E^1_{s,t} = h^S_{s+t}(\underline{F}_s,\underline{F}_{s-1}) \approx h_t(Q^s X,\partial)$ (compare p. 110-112 [M6]). Thus there is no ζ_o needed in the formula.

We will write $E^r_{s,t}(X,\Lambda)$ for $E^r_{s,t}(X,H_*(\ ;\Lambda))$. For $\Lambda =$ Z/p we note that d^1: $E^1_{t,1} \to E^1_{t,o}$ corresponds to ρ_*: $H_t(QX,\eta X;Z/p) \to H_t(X;Z/p)$. An element on the left is a formal polynomial in formal Dyer Lashof operations on classes of $H_*(X;Z/p)$. The differential is evaluation. For example if $x,y \in H_1(X;Z/2)$ and $Q^2 z \in H_3(X;Z/2)$, then $d^1(Q^3[xy]\cdot Q^6[Q^2 z]) = (\Sigma Q^{3-i}xQ^i y)(Q^6Q^2 z) = (x^2Q^2y + Q^2xy^2)Q^5Q^3 z$ using the Cartan formula, unstability and the Adem relation $Q^6Q^2 = Q^5Q^3$.

The cohomology spectral sequence $E_1^{s,t}(X,h^*) = h^t(Q^sX,\partial)$ converges to $h_S^*(X)$. This is related to the homology spectral sequence by the usual duality theorem and the universal coefficient theorems if $h^* = H^*(\ ;\Lambda)$.

The existence of such a spectral sequence was noted by P. May ([M6], p. 155) and D. W. Anderson [A2]. Using completely different methods, Haynes Miller defined a delooping spectral sequence and computed $E_{s,t}^2(X;Z/2)$ [M8]. Miller was able to compute the spectral sequence in certain cases and give some applications. We had discovered this spectral sequence independently after realizing that a certain computation was a d^3 in some spectral sequence and then identifying the spectral sequence.

In a forthcoming paper we will show that the spectral sequences above are equivalent. We will also describe $E^2(X;Z/p)$ and do a number of computations. We expect that there will be many applications of this spectral sequence to infinite loop space theory and stable homology theory.

§5. Cycles and Q_k maps

Let X be a Q algebra. The Eilenberg Moore spectral sequence has $E_2^{*,*}$ equal to a functor of $H^*(X;\Lambda)$ as a Λ coalgebra and converges to $H^*(BX;\Lambda)$. If $x \in H^t(X;\Lambda)$ is represented by a map $f: X \to K(\Lambda,t)$, then f is a loop map, i.e., $f \simeq \Omega g$ for $g: BX \to K(\Lambda,t+1)$, if and only if x is an infinite cycle, i.e., it represents a class in $H^*(BX;\Lambda)$. Moreover, x is a d_k cycle if and only if f is an A_k map [S4].

Similarly, f is an infinite loop map if and only if x is an infinite cycle in the Miller delooping spectral sequence. For in that case, x survives to a stable class $y \in H_S^t(X;\Lambda)$. We prove that x is a d_k cycle if and only if f is a Q_k map.

<u>Theorem 5.1</u>. Let $x \in h^t(X) = [X,K_t] = E_1^{o,t}$ be represented by a map $f: X \to K_t$. Then f is a Q_k map if and only if x is a k-cycle.

Proof: The class x is a k-cycle if and only if there is a class $x_k \in h_S^t(\underline{F_k})$ such that $(i^k)^* x_k = x \in h_S^t(\underline{F_o}) = h^t(X)$ (see 4.6 and Ch. XI [M1]). Thus x is a k cycle if and only if there is a map $G_k: \underline{F_k} \to \underline{K_t}$ such that $i^k G_k g$ is adjoint to f. This means that f is a Q_k map by Theorem 3.2.

<u>Corollary 5.2</u>. Let $f: X \to K_t$ be a map between Q algebras. Then f is homotopic to an infinite loop map if and only if $[f] \in E_1^{o,t} = [X,K_t]$ is an infinite cycle. Furthermore [f] survives to $E_2^{o,t}$ if and only if f is transfer commuting, i.e. a Q_1 map.

If X is an infinite loop space then the Eilenberg Moore spectral sequence $E_r^{*,*} \Longrightarrow H^*(BX)$ is in the category of abelian Hopf algebras. This implies that if $d_r(x) \neq 0$ then $r = p^{k+1}-1$ or $2p^k-1$ (compare Theorem 2.4 [K]). It can be shown that if $d_r(x) \neq 0$ for r as above in the Eilenberg Moore spectral sequence, then $d_k(x) \neq 0$ in the Miller spectral sequence. Thus if there is an obstruction to $f: X \to K(Z/p,n)$ being an $A_p k$ map, there is also an obstruction to f being a Q_k map.

In Theorem A [MST] it was shown that every transfer commuting endomorphism of BSO was homotopic to a stable, i.e., infinite loop, map. This result implies that $E_2^{o,t} = E_\infty^{o,t}$ in the spectral sequence converging to $[bso_{(p)}, bso_{(p)}]$.

Assume now that $[f] \in h^t(X) = [X,K_t]$ is a 2 cycle and let $E \xrightarrow{\Pi} X \xrightarrow{f} K_t$ be the induced fibration. Then by Theorems 3.3 and 2.5, E is a transfer space. Thus to construct a transfer counterexample, we need to be able to compute differentials in the spectral sequence.

Let X be the stable 2 stage Postnikov system with k invariant $P^{p^2 n}P^{pn}P^{n}\iota: K(Z/p,2n+1) \to K(Z/p,2np^3+1)$ for $n \not\equiv -1(p)$. Using techniques developed in [K], we can show that there is a class $\psi \in H^{2(n+1)p^3-2}(X;Z/p)$ which does indeed represent a 2 cycle but not a 3 cycle. Moreover an elementary Postnikov system argument proves that the fiber induced by ψ is not an infinite loop space. Indeed ψ is the p^3 transpotence of the fundamental class of BX. Thus this fiber is a counterexample to the transfer conjecture.

Unfortunately, the proof of this fact requires a fairly complete description of $E^2_{s,t}(X)$ and $E^\infty_{s,t}(X)$. Results of Madsen and Snaith [M4] concerning the transfer conjecture were recently brought to our attention. We extend their results to find a more "real life" counterexample. To establish this counterexample we need much less information about the E^2 term and no explicit knowledge of E^∞. Instead we need p torsion results on $E^r_{s,t}(X;Z_{(p)})$.

§6. Partial computation of E^2

Assume that X is a connected Q algebra of finite type. Since X is an infinite loop space, $H_*(X;Z/p)$ is an abelian Hopf algebra. Borel's theorem implies that $H_*(X;Z/p)$ is a free commutative algebra modulo relations of the form y^{p^k}. To simplify the following arguments we will make the strong assumption that $H_*(X;Z/p)$ is the free algebra on a Z/p module M with basis $\{y_j\}$. The examples we use to construct the counterexample are $X = BU$ and $X = BSU$, which satisfy this hypothesis.

Definition 6.1. For a graded connected Z/p module N with basis $\{x_j\}$, let AN be the underlying module of the free commutative Z/p algebra on N. Let TN denote the Z/p module with basis $Q^I x_j$, where Q^I is an admissible Dyer Lashof operation of excess greater than the degree of x_j.

Theorem 6.2. If $H_*(X;Z/p)$ is the free commutative algebra on a module M, then

$$H_*(Q^s X;Z/p) = (AT)^s AM$$

$$= ATA\ldots TAM.$$

Proof: This follows easily from p. 42 [M7].

Thus $H_t(Q^s X;Z/p)$ is generated by monomials in Dyer Lashof operations on monomials in Dyer Lashof operations on ... monomials in basis elements $\{x_j\}$ of $M \approx QH_*(X;Z/p)$. For $u \in H_t(Q^{s-1}X;Z/p)$ we write $1(u) = Q^{\emptyset}(u)$ for $\eta_*(u)$ in $H_t(Q^s X;Z/p)$.

Example 6.3. $\alpha = Q^6\{Q^2(x)1(y)\}Q^7Q^4\{Q^1(z)\}$ is an element of $H_{23}(Q^2X;Z/2)$ where $x,y,z \in H_1(X;Z/2)$.

The $E^1_{s,t}$ term of the Miller spectral sequence is the quotient of $(AT)^s AM \approx H_*(Q^s X;Z/p)$ by the image of the degeneracies $S_i: Q^{s-1}X \to Q^s X$ of (1.4). The differential of Theorem 4.5 extends to $d = \sum_{i=0}^{s} (-1)^i d_i$:

$(AT)^s AM \to (AT)^{s-1} AM$ for $s > 0$. Note that the map d_o is given by the composite

$$H_t(Q^S X) \underset{\sim}{\overset{\Sigma}{\to}} H_t^S(\Sigma^\infty Q^s X) \overset{\sigma_*}{\to} H_t^S(\Sigma^\infty Q^{s-1} X) \underset{\sim}{\overset{\Sigma^{-1}}{\to}} H_t(Q^{s-1} X).$$

Thus $(AT)^* AM$ is the unnormalized version of $E^1_{*,*}$ and so the two are chain equivalent.

The complex $(AT)^* AM$ is much too large and complicated to use effectively. Consider the subcomplex $T^* M = \{T^s M\}$. We may write a generator of $T^s M$ as $Q^{I_1}|\ldots|Q^{I_s}|x$ where x is a generator of M and I_j is an admissible sequence of excess greater than $\deg I_{j+1} + \ldots + \deg I_s + \dim x$. The face map d_j removes the jth bar. If $i = 0$ we get 0 unless $Q^{I_1} = Q^\emptyset = 1$ in which case the first term is omitted. Thus $T^* M$ is an unstable unnormalized bar construction for the R module $M \approx QH_*(X;Z/p)$.

<u>Theorem 6.4</u>. The inclusion of the subcomplex $T^* M$ into $(AT)^* AM$ is a chain equivalence.

<u>Proof</u>. Our method is to successively contract out the algebra structure of $(AT)^s AM$. First define a filtration of $(AT)^* AM$, $T^s M \approx F_o^s \subset F_1^s \subset \ldots \subset F_s^s \subset F_{s+1}^s = (AT)^s AM$, by setting $F_k^s = \mathrm{Im}[(AT)^k T^{s-k} M \to (AT)^s AM]$ for $0 \le k \le s$. Recall that $d_i = (AT)^{i-1} \mu (AT)^{s-i}$ where $\mu: ATAT \longrightarrow AT$ is evaluation. Thus $d_i(F_k^s) \subset F_{k-1}^{s-1}$ if $i < k$ and $d_i(F_k^s) \subset F_k^{s-1}$ if $i \ge k$.

We will show that the quotient complex F_k^*/F_{k-1}^* is acyclic for $k \ge 1$. Thus $F_{k-1}^* \longrightarrow F_k^*$ is a chain equivalence and the theorem follows by iteration. Alternatively the filtration gives rise to a spectral sequence converging to $H_*((AT)^* AM)$. We show that $E^2_{*,t} \approx H_*(F_t^*, F_{t-1}^*) = 0$ for $t \neq 0$ and thus $H_*(T^* M) \approx E^2_{*,o} = E^\infty_{*,o} \approx H_*((AT)^* AM)$.

By Definition 6.1. AN is the module with generators of the form $x_1 \ldots x_n$ where the x_i's are generators of N. Also there are generators $1(x_i) = Q^{\emptyset}(x_i)$ in TN. Define a homomorphism $c: AN \longrightarrow ATN$ by $c(x_1 \ldots x_n) = 1(x_1) \ldots 1(x_n)$. Letting $N = T^{s-k+1}M$ we have an extension

$$c_k = (AT)^{k-1}c : F_k^s \longrightarrow F_k^{s+1}$$

for $k > 0$. Since $\mu(1(x_1) \ldots 1(x_n)) = x_1 \ldots x_n$, $d_k c_k(x_1 \ldots x_n) = x_1 \ldots x_n$ M reover $d_i c_k = c_k d_{i-1}$ if $i > k$. Since $d_i(F_k^{s+1}) \subset F_{k-1}^{s+1}$ for $i < k$, c_k extends to a homomorphism

$$c' : F_k^s / F_{k-1}^s \longrightarrow F_k^{s+1} / F_{k-1}^{s+1}$$

such that $dc' - c'd = 1$. Thus c' is a contradiction and so F_k^* / F_{k-1}^* is acyclic for $k > 0$ as required.

In an attempt to make the formula $dc' - c'd = 1$ more comprehensible, we carry out the necessary computations for the element $\alpha \in \mathrm{Im}(ATATM) = F_2^2$ of Example 6.3.

$$d_0\alpha = 0$$

$$d_1\alpha = \sum_t (Q^{6-t}Q^2(x)Q^t(y))(Q^7Q^4Q^1(z))$$

$$= 0$$ (Here we use the Cartan formula, excess, and the Adem relations $Q^7Q^4Q^1 = Q^7Q^3Q^2 = 0$.)

$$d_2\alpha = Q^6\{(Q^2x)(y)\}Q^7Q^4\{z^2\}$$ (Here we use $Q^1z = z^2$.)

$$c(\alpha) = Q^6\{1[Q^2(x)]1[1(y)]\}\ Q^7Q^4\{1[Q^1(z)]\}$$

$$d_0c(\alpha) = 0$$

$$d_1c(\alpha) = (\sum_t Q^{6-t}[Q^2(x)]Q^t[1(y)])\ Q^7Q^4[Q^1(z)] \in F_1^2$$

$$d_2c(\alpha) = \alpha$$

$$d_3c(\alpha) = Q^6\{1[Q^2x]1[y]\}\ Q^7Q^4\{1[z^2]\} = c(d_2\alpha).$$

A standard argument will show that T^sM is chain equivalent to $\tilde{T}^sM$ where $\tilde{T}M = TM/(1M)$ is the normalized version. Thus a generator of $\tilde{T}^sM$ may be written $Q^{I_1}|\ldots|Q^{I_s}|x_j$ where x_j is a generator of M, I_j is admissible and nontrivial ($I_j \neq \emptyset$), and excess of I_j is greater than the degree of $Q^{I_{j+1}}\ldots Q^{I_s}x_j$. This theorem implies that $E^1_{s,t}$ is chain equivalent to $\tilde{T}^sM$. In order to form $E^2_{s,t}$ we must still take into account the Adem relations.

Miller [M8] started with essentially the complex $\tilde{T}^sM$. He described $E^2_{s,t}$ as an unstable Tor functor on the Dyer Lashof algebra and computed $E^2_{s,t}(X,Z/2)$ when the Dyer Lashof action on $H_*(X;Z/2)$ is trivial. Thus the Miller spectral sequence is analogous to an unstable Adams spectral sequence.

Remark 6.5. If $H_*(X;Z/p)$ is not a free algebra, but instead there are relations $y^{p^{k+1}} = 0$ for dim $y = 2n$, then we must add the generators $Q^{I_1}|\ldots|Q^{I_{s-1}}|Q_o^k|y$ to the above collection where $Q_o^k = Q^{p^k n}\ldots Q^{pn}Q^n$. This situation will be fully discussed in our forthcoming paper.

§7. p torsion in $E^r_{s,t}$

If X is an infinite loop space, then it is possible to define homology Pontrjagin-Thomas p^{th} power operations

$$\mathfrak{P} : H_{2n}(X;Z/p^{k-1}) \to H_{2np}(X;Z/p^k) \to ([M3],[M7])$$

Let β_k be the Bockstein operator associated with

$$0 \to Z/p \to Z/p^{k+1} \xrightarrow{r} Z/p^k \to 0. \quad \text{Then}$$

(7.1)

a) $r_* \mathfrak{P}(x) = x^p$

b) $\beta_k \mathfrak{P}(x) = x^{p-1}\beta_{k-1}x$ if $p > 2$ or $k > 1$

$= x\beta x + Q^{2n}\beta x$ if $p = 2$ and $k = 1$.

If all of the higher p torsion of X arises from Pontrjagin products, then X is called Henselian. More precisely let $\tilde{\beta}_k$ be the Bockstein operator associated with $0 \to Z_{(p)} \xrightarrow{p^k} Z_{(p)} \xrightarrow{r} Z/p^k \to 0$. Note that $r\tilde{\beta}_k = \beta_k$ if r is the reduction $Z_{(p)} \to Z/p$. Then X is Henselian if the p^k torsion of $H_*(X;Z_{(p)})$ for $k > 1$ is generated by elements of the form $\beta_k \mathfrak{P}\ldots\mathfrak{P}(x)$ for $x \in H_*(X;Z/p)$ (compare definition 1.7 [M3]).

Theorem 7.2. If $H_*(X;Z_{(p)})$ has no p^2 torsion, then QX is Henselian at p.

Proof: See p. 63 [M7].

Note that if QX is Henselian, then the mod p reduction of higher torsion is decomposable, unless p=2 and k=2. Since $E^2_{1\,*}(X;Z_{(p)})$ is a subquotient of $H_*(QX;Z_{(p)})$ in which decomposables have been contracted out, by Theorem 6.4, we may expect that there is no higher torsion in that group. Indeed more is true.

Theorem 7.3. Assume that $H_*(X;Z/p)$ is a polynomial algebra and that $H_*(X;Z_{(p)})$ contains no p^2 torsion. Then $E^r_{s,t}(X;Z_{(p)})$ and $E^{s,t}_r(X;Z_{(p)})$ are Z/p modules for $r \geq 2$ and $s \geq 1$.

Proof: By the universal coefficient theorem and the fact that the homology of a Z/p module is a Z/p module, it suffices to prove that $E^2_{s,t}$ is a Z/p module for $s \geq 1$. From the Bockstein spectral sequence for $H_*(QX)$ (p. 48 [M7]), the infinite factors of $H_*(QX;Z_{(p)})$ are in the image of $\eta: X \to QX$ or arise from formal products of generators of infinite factors of $H_*(X;Z_{(p)})$ such as $1(x)1(y)$. But such elements are either in the image of the degeneracies $Q^i\eta Q^j: Q^{s-1}X \to Q^sX$ or they are decomposables. Moreover if there were an infinite factor in $E^2_{s,t}(X;Z_{(p)})$, then it would reduce non trivially to $E^2_{s,t}(X;Z/p)$. Since $E^2_{s,t}(X;Z/p)$ is the homology of $\tilde{T}^*M$, in which degeneracies and decomposables are divided out, there is a contradiction.

Similarly if $z \in E^2_{s,t}(X;Z_{(p)})$ generated a p^k factor for $k \geq 2$, then it must be degenerate or of the form $\beta_k\mathcal{P}(y)$. If $p > 2$ or $k > 2$ then by (7.1), z is represented by a degenerate or a decomposable in $E^2(X;Z/p)$ and so we reach a contradiction again. Finally if $p=2$ and $k=2$, then the 4 torsion element is represented by $Q^{2n}\beta x$ in $E^2_{s,t}(X;Z/2)$. But this element is in the d^1 image of $Q^{2n}[\beta x]$ modulo terms of lower filtration. Thus $Q^{2n}\beta x$ cannot represent a nonzero element in $E^2_{s,t}(X;Z/2)$, and the proof is complete.

Note that while $E^{s,t}_r(X;Z_{(p)})$ is a Z/p module for $s > 0$, the edge homomorphism

$$E^{o,t}_r \to H^t(X;Z_{(p)})$$

is a monomorphism. Thus the edge term will often have infinite factors.

<u>Corollary 7.4</u>. Assume that $H_*(X;Z/p)$ is a polynomial algebra and that $H_*(X;Z_{(p)})$ has no p^2 torsion. Let $c \in E_2^{o,t}(X;Z_{(p)})$. Then $p^r c$ is an $r+1$ cycle for all $r \geq 1$.

Proof: Assume that $p^{r-1}c$ is an r cycle and that $d^{r+1}(p^{r-1}c) = y$. Then $d^{r+1}(p^r c) = py = o$.

§8. The counterexample

In this section all spaces will be localized at 2 and $H^*(\ )$ will mean $H^*(\ ;Z_{(2)})$. We will write a class $\zeta \in H^n(X)$ and a representing map $\zeta: X \to K(Z_{(2)},n)$ interchangeably.

Recall that $H_*(BU) \approx Z_{(2)}[a_1,a_2,\ldots]$ as algebras [L3] and so the hypotheses of Corollary 7.4 are satisfied. Also $H^*(BU) \approx Z_{(2)}[c_1,c_2,\ldots]$ as algebras where c_k is the Chern class of dimension 2k. Let $S_k \in PH^{2k}(BU) \approx Z_{(2)}$ be the primitive class dual to a_k in the basis of monomials. We may express S_k as the Newton polynomial kc_k + decomposables (Chapter IV [L3]).

<u>Theorem 8.1</u>. For each k, $4S_k: BU \to K(Z_{(2)},2k)$ is a Q_2 map and its induced homotopy fiber E_k is a transfer space.

Proof: Madsen [M4] has shown that $2S_k$ is a transfer commuting or Q_1 map. By Corollary 5.2 $[2S_k]$ represents a nonzero class in $E_2^{o,2k}$. By Corollary 7.4, the class $4S_k$ is a d_2 cycle and so by Theorem 5.1, the map $4S_k$ is a Q_2 map. Finally, Theorem 3.3 implies that E_k is a Q_2 or transfer space.

In summary, $[\ ,E_k]$ is a representable homotopy functor which admits a transfer. Let $\alpha(k)$ be the number of 1's in the diadic expansion of k. It follows from work of Adams [A1] (see also [M4]) that 2^nS_k is a stable class if and only if $n \geq \alpha(k)-1$. If $\alpha(k) \leq 3$ then $4S_k: BU \to K(Z_{(2)},2k)$ may be taken to be an infinite loop map and so $[\ ,E_k]$ extends to a cohomology theory. However if $\alpha(k) \geq 4$, we then will get a counterexample to the transfer conjecture.

<u>Theorem 8.2</u>. The fiber $E = E_{15}$ of $4S_{15}$ has a Q_2 structure which does not extend to a Q_∞ structure. Thus $[\ ,E]$ is a counterexample to the transfer conjecture.

By Proposition 2.2, the Q_2 structure on E determines its H structure. Thus it suffices to show that there is no infinite loop space which is H equivalent to E. In fact, we will show that there is no H space F such that $H_*(\Omega^2 F) \approx H_*(E)$ as algebras.

We first outline the proof. Assume to the contrary that such an F existed. Then we will show that a Postnikov approximation of F fibers over a Postnikov approximation of BSU. Moreover, it will be induced by a map $\tau\colon BSU \to K(Z_{(2)},32)$ with $\Omega^2\tau \simeq 4S_{15}$. However, if t_{16} is the generator of $QH^{32}(BSU)$, then we will show that $\tau = 4t_{16}$ modulo decomposables whereas $PH^{32}(BSU)$ is generated by $8t_{16}$ modulo decomposables. This will imply that τ cannot be chosen to be primitive and that F will not in fact have an H structure.

We first record some classical facts about BSU.

Lemma 8.3. $H^*(BSU) \simeq Z_{(2)}[t_2,t_3,\ldots]$ as algebras. If $\sigma^2\colon QH^{2k+2}(BSU) \to PH^{2k}(BU)$ is the 2-fold loop suspension followed by the identification $\Omega^2 BSU \simeq BU$, then $\sigma^2 t_{k+1} = S_k$. Finally $PH^{32}(BSU)$ is generated by a class γ which equals $8t_{16}$ modulo decomposables.

Proof: These results can be found in the literature (e.g. [L3], [S3]). The first is classical. The second follows from the collapse of the Eilenberg-Moore spectral sequences ${}_1E_2^{**} = \mathrm{Tor}_{H^*(BSU)}(Z_{(2)},Z_{(2)}) \Longrightarrow H^*(\Omega BSU) \approx H^*(SU)$ and ${}_2E_2^{**} = \mathrm{Tor}_{H^*(SU)}(Z_{(2)},Z_{(2)}) \Longrightarrow H^*(\Omega SU) \approx H^*(BU)$.

To see the last result recall [L3] that the Newton polynomial $S_{16} = 16c_{16} + 2D + 2c_2^8 + c_1^{16}$ generates $PH^{32}(BU)$ for D a decomposable. If we replace c_k by t_k for $k > 1$ and 0 for $k = 1$ in the above polynomial, then we get a primitive $2\gamma = 16t_{16} + \cdots + 2t_2^8$ in $H^{32}(BSU)$. But γ is primitive and not divisible by 2 and so generates $PH^{32}(BSU)$.

For a simply connected H space X, there is a Postnikov decomposition $X \to \ldots \to X^n \xrightarrow{\pi} X^{n-1} \to \ldots \to X_1 = *$. In particular $\pi: X^n \to X^{n-1}$ is a principle fibration induced by an H map $k^n: X^{n-1} \to K(\pi_n(X), n+1)$. We will assume knowledge of the Postnikov systems for BU and BSU (see §2 [AP]).

Lemma 8.4. Assume that F is an H space such that $H^*(\Omega^2 F) \simeq H^*(E)$ as Hopf algebras. Then $F^n = BSU^n$ for $n < 32$.

Proof: Note that $\pi_k(F) \simeq \pi_{k-2}(E) \simeq \pi_{k-2}(BU) \simeq \pi_k(BSU)$ for $k < 32$. Thus $F^4 = F^5 = K(Z_{(2)}, 4)$. Also $F^6 = F^7$ is the 2-stage Postnikov system with k invariant $k: K(Z_{(2)}, 4) \to K(Z_{(2)}, 7)$. But $H^7(K(Z_{(2)}, 4); Z_{(2)}) \simeq Z_{(2)}$ is generated by $\tilde{\beta} Sq^2 \iota_4$ where $\tilde{\beta}$ is the Bockstein associated with $0 \to Z_{(2)} \to Z_{(2)} \to Z/2 \to 0$.

Thus the k invariant for F^6 is $v\tilde{\beta} Sq^2 \iota$ for $v \in Z_{(2)}$ and so the k invariant for ΩF^6 is $v(\iota_3)^2$. Moreover the k invariant for $\Omega^2 F^6 = E^4$ is 0 and so $\Omega^2 F^6 \simeq K(Z_{(2)}, 2) \times K(Z_{(2)}, 4)$. The H structure on $\Omega^2 F^6$ depends on v. More precisely

$$\Delta \iota_4 = \iota_4 \otimes 1 + v(\iota_2 \otimes \iota_2) + 1 \otimes \iota_4 .$$

Since $\Delta c_2 = c_2 \otimes 1 + c_1 \otimes c_1 + 1 \otimes c_2$ in $H^*(BU)$ and so in $H^*(E^4)$, we may assume that $v = 1$. Since the first k invariant for BSU is $\tilde{\beta} Sq^2 \iota$ ([AP], [S3]), we have $F^6 = BSU^6$.

Inductively assume that $F^{2n-2} = F^{2n-1} = BSU^{2n-1}$ for $2n < 32$. Then the k invariant k^{2n} for F^{2n} is in $PH^{2n+1}(BSU^{2n-1})$. It is not hard to compute that this group is $Z_{(2)}$ and is generated by a class x with $j^* x = \tilde{\beta} Sq^2 \iota_{2n-2} \in H^{2n+1}(K(Z_{(2)}, 2n-2), Z_{(2)})$. Since $\sigma^2(k^{2n})$ is the k invariant for $E^{2n-2} = BU^{2n-2}$, knowledge of the k invariants for BU and BSU implies that $F^{2n} = BSU^{2n}$.

Using these results it follows that F^{32} appears in the following diagram of induced fibrations.

$$\begin{array}{ccccc} & & F^{32} & & \\ & & \downarrow & & \\ K(Z_{(2)},32) & \xrightarrow{j} & BSU^{32} & \xrightarrow{\tau} & K(Z_{(2)},32) \\ & & \downarrow \pi & & \\ K(Z_{(2)},30) & \xrightarrow{j} & BSU^{30} & \xrightarrow{k} & K(Z_{(2)},33) \end{array}$$

where j denotes the fiber inclusion, $j^*(k) = \tilde{\beta}Sq^2\iota$ and $\sigma^2(\tau) = 4S_{15}$.

This is no longer a Postnikov tower since dim k > dim τ. Let λ be determined by $j^*(\tau) = \lambda\iota \in H^{32}(K(Z_{(2)},32),Z_{(2)}) \approx Z_{(2)}$. (It can be shown that $\lambda = 4\cdot 15!$). The final stage of the Postnikov system is thus

$$\begin{array}{ccccc} K(Z/\lambda,31) & \xrightarrow{j} & F^{32} & & \\ & & \downarrow & & \\ K(Z_{(2)},30) & \xrightarrow{j} & BSU^{30} & \xrightarrow{k'} & K(Z/\lambda,32). \end{array}$$

Moreover $r_2j^*(k') = Sq^2\iota$ in $H^{32}(K(Z_{(2)},30);Z/2)$ and $\pi^*(k') = r_\lambda(\tau) \in$ $\pi^*(k') = r_\lambda(\tau) \in H^{32}(BSU^{32};Z/\lambda)$ where r is the appropriate reduction homomorphism and π: $BSU^{32} \to BSU^{30}$. Q.E.D.

To finish the proof of Theorem 8.2, it suffices to show that k' cannot be chosen to be a primitive, for then F^{32} and thus F would not be an H space. Since $\sigma^2(\tau) = 4S_{15}$, Lemma 8.3 implies that τ cannot be chosen to be primitive. In fact $\Delta\tau$ contains the term $t_2^8 \otimes t_2^8$. Thus $\Delta r_\lambda(\tau)$ contains a nonzero middle term. Since π is an H map, k' cannot be primitive and the proof is complete.

Several remarks are in order about this example. First we are not claiming that E has no infinite loop structure. Indeed it is possible that the $\otimes$ infinite loop structure on BU may induce some Q_∞ structure on E. If, however, we consider the fiber E' of $4S_{30}$: $BSU \to K(Z_{(2)},60)$, then using the uniqueness of the infinite loop structure on BSU [AP] it is possible to prove that E' has no infinite loop structure.

Corollary 5.2 implies that $[4S_{15}] \in E_r^{0,30}$ is not an infinite cycle. The class $\alpha = \tilde{\beta}(Q^{16}|Q^8|Q^4|a) \in H_{27}(Q^3BU,\partial)$ represents an element $[\alpha] \in E^3_{3,27}$. It can be shown that $< d_3(4S_{15}),\alpha > \neq 0$ under the pairing of the torsion submodules of $H^{28}(Q^3BU)$ and $H_{27}(Q^3BU)$. On the other hand, for dimension reasons $8S_{15}$ will be an infinite cycle. Its mod 2 reduction can be shown to represent $Sq^{16}Sq^8Sq^4\iota$ in $H_S^{30}(bu;Z/2) \simeq \Sigma^2A/A(Sq^1,Sq^3)$. The Miller spectral sequence for $H_*(BU;Z/p)$ and $H_*(BO;Z/p)$ will be completely analyzed in our forthcoming paper.

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