SYMMETRIZATION OF BRACE ALGEBRAS

MARILYN DAILY AND TOM LADA

ABSTRACT. We show that the symmetrization of a brace algebra structure yields the structure of a symmetric brace algebra. We also show that the symmetrization of the natural brace structure on $\bigoplus_{k\geq 1} Hom(V^{\otimes k}, V)$ coincides with the natural symmetric brace structure on $\bigoplus_{k\geq 1} Hom(V^{\otimes k}, V)^{as}$, the space of antisymmetric maps $V^{\otimes K} \to V$.

1. INTRODUCTION

Brace algebras were first studied in the context of multilinear operations on the Hochschild complex of an associative algebra [3, 2, 1]. Symmetric brace algebras, in which the brace operations possess the property of graded symmetry, were subsequently introduced in [5]. Just as one may construct L_∞ algebra structures by anti (skew) symmetrizing A_∞ algebra structures [4], we show in this note that the symmetrization of a brace algebra structure yields a symmetric brace algebra structure. We prove in Section 5 that

$$f\langle g_1, \ldots, g_n\rangle := \sum_{\sigma\in S_n} \epsilon(\sigma) f\{g_{\sigma(1)}, \ldots, g_{\sigma(n)}\}$$

where $\langle\, , \,\rangle$ and $\{\, , \,\}$ denote symmetric and non symmetric braces respectively.

The motivating example of a brace algebra is $\bigoplus_{k\geq 1} Hom(V^{\otimes k}, V)$, and the fundamental example of a symmetric brace algebra is the subspace of anti symmetric maps, $\bigoplus_{k\geq 1} Hom(V^{\otimes k}, V)^{as}$.

In Section 6, we show that these algebras are related by

$$\sum_{\sigma\in S_n} \epsilon(\sigma) as(f\{g_{\sigma(1)}, \ldots, g_{\sigma(n)}\}) = as(f)\langle as(g_1), \ldots, as(g_n)\rangle,$$

where $as(f)(v_1, \ldots, v_k) := \sum_{\sigma\in S_k} (-1)^\sigma \epsilon(\sigma) f(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$ and $\epsilon(\sigma)$ is just the Koszul sign of the permutation.

In Sections 2 and 3, we review the definitions and fundamental examples of brace algebras and symmetric brace algebras respectively. Section 4 contains a collection of technical lemmas that are needed to prove the main theorems in the final two sections.

2. BRACE ALGEBRAS

Definition 1. A *brace structure* on a graded vector space consists of a collection of degree 0 multilinear braces $x, x_1, \ldots, x_n \mapsto x\{x_1, \ldots x_n\}$ which satisfy the identity, $x\{\,\} = x$, and in which $x\{x_1,\ldots,x_n\}\{y_1,\ldots,y_r\}$ is equal to

$$\sum \epsilon \cdot x\{y_1,\ldots,y_{i_1}, x_1\{y_{i_1+1},\ldots,y_{j_1}\}, y_{j_1+1},\ldots,y_{i_n}, x_n\{y_{i_n+1},\ldots,y_{j_n}\}, y_{j_n+1},\ldots,y_r\}.$$

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In the above formula, the sum is over all sequences $0 \leq i_1 \leq j_1 \leq \ldots \leq i_n \leq j_n \leq r$, and ϵ is the Koszul sign of the permutation which maps $(x_1,..., x_n, y_1,..., y_r)$ to

$$(y_1,..., y_{i_1}, x_1, y_{i_1+1},..., y_{j_1}, y_{j_1+1},..., y_{i_n}, x_n, y_{i_n+1},..., y_{j_n}, y_{j_n+1},..., y_r).$$

The motivating example for a brace algebra structure is the space $Hom(V^{\otimes N}, V)$ with the natural brace operation of degree $-n$ given by the composition

$$f\{g_1, \ldots, g_n\} = \sum_{k_0+\cdots+k_n=N-n} f\left(1^{\otimes k_0} \otimes g_1 \otimes 1^{\otimes k_1} \otimes \cdots \otimes g_n \otimes 1^{\otimes k_n}\right),$$

where $f \in Hom(V^{\otimes N}, V)$. This operation arises from the endomorphism operad of V considered in [1]. This operation was also utilized in the context of the Hochschild complex of the associative algebra V in [3] and [2]. After a regrading, this example may be regarded as a special case of the following

Example 2. Let V be a graded vector space and consider the graded vector space $B_*(V)$ where

$$B_s(V) := \bigoplus_{p-k+1=s} Hom(V^{\otimes k}, V)_p$$

and where $Hom(V^{\otimes k}, V)_p$ denotes the space of k-multilinear maps of degree p. Given $f \in Hom(V^{\otimes N}, V)_p$ and $g_i \in Hom(V^{\otimes a_i}, V)_{q_i}$, define $f\{g_1, \ldots, g_n\} \in Hom(V^{\otimes r}, V)_{p+q_1+\cdots+q_n}$ where $r = a_1 + \cdots + a_n + N - n$ by

$$f\{g_1,..., g_n\} = \sum_{k_0+\cdots+k_n=N-n} (-1)^\beta f(1^{\otimes k_0} \otimes g_1 \otimes 1^{\otimes k_1} \otimes \cdots \otimes 1^{\otimes k_{n-1}} \otimes g_n \otimes 1^{\otimes k_n}),$$

where

$$\beta = \sum_{j<i} [a_i - 1]\,[k_j + a_j] + \sum_i (N - i)\, q_i + \sum_{j<i} q_i\, a_j.$$

Remark 3. In Example 2, suppose that there exists a collection of maps

$$\mu_k \in Hom(V^{\otimes k}, V)_{k-2} \in B_{-1}(V).$$

If we let $\mu = \mu_1 + \mu_2 + \ldots$, then an A_∞ algebra structure on V may be described by the brace relation $\mu\{\mu\} = 0$ [5].

3. Symmetric Brace Algebras

Definition 4. An *n-unshuffle* of N elements is a partition $\sum_{i=1}^n a_i = N$ and a permutation $\gamma \in S_N$ such that

$$\gamma(1) < \cdots < \gamma(a_1), \gamma(1 + a_1) < \cdots < \gamma(a_2 + a_1), \ldots, \gamma\left(1 + \sum_{i=1}^{n-1} a_i\right) < \cdots < \gamma(N).$$

Definition 5. A *symmetric brace algebra* is a graded vector space together with a collection of degree zero multilinear braces $f\langle g_1, \ldots, g_n\rangle$ which are graded symmetric in $g_1, \ldots, g_n$. In a symmetric brace algebra, it is also required that $f\langle\rangle = f$, and that $f\langle g_1, \ldots, g_n\rangle\langle x_1, \ldots, x_r\rangle$ be equal to

$$\sum_{\substack{\gamma \text{ is } (n+1) \\ \text{unshuffle}}} \epsilon \cdot f\Big\langle g_1\big\langle x_{\gamma(1)},..., x_{\gamma(a_1)}\big\rangle,..., g_n\big\langle x_{\gamma\left(1+\sum_{i=1}^{n-1} a_i\right)},..., x_{\gamma\left(\sum_{i=1}^{n} a_i\right)}\big\rangle, x_{\gamma\left(1+\sum_{i=1}^{n} a_i\right)},..., x_{\gamma(r)}\Big\rangle,$$

where ϵ is the Koszul sign of the permutation which maps $(g_1, ..., g_n, x_1, ..., x_r)$ to

$$\left(g_1, x_{\gamma(1)}, \dots, x_{\gamma(a_1)}, g_2, \dots, x_{\gamma\left(1+\sum_{i=1}^{n-1} a_i\right)}, \dots, x_{\gamma\left(\sum_{i=1}^{n} a_i\right)}, g_n, x_{\gamma\left(1+\sum_{i=1}^{n} a_i\right)}, \dots, x_{\gamma(r)}\right).$$

Just as with brace algebras, the fundamental example of a symmetric brace algebra is provided by the space of antisymmetric maps of degree p, $Hom(V^{\otimes k}, V)_p^{as}$. To be precise, we have

Example 6. Let V be a graded vector space and $B_*(V)$ be the graded vector space given by

$$B_s(V) = \bigoplus_{p-k+1=s} Hom(V^{\otimes k}, V)_p^{as},$$

Given $f \in Hom(V^{\otimes k}, V)_p^{as}$ and $g_i \in Hom(V^{\otimes a_i}, V)_{q_i}^{as}$, $1 \leq i \leq n$, define the symmetric brace

$$f\langle g_1, \dots, g_n\rangle(x_1, \dots, x_r) = (-1)^{\delta} \sum_{\substack{\gamma \text{ is an} \\ (a_1|a_2|\dots|a_{n+1}) \\ \text{unshuffle}}} \chi(\gamma) f(g_1 \otimes \cdots \otimes g_n \otimes 1^{\otimes N-n})(x_{\gamma(1)}, \dots, x_{\gamma(r)}),$$

where

$$\delta = \sum_i^n (N-i) q_i + \sum_{j<i} q_i a_j + \sum_{j<i} a_i a_j + \sum_i (n-i) a_i,$$

and χ is the antisymmetric Koszul sign of the permutation γ.

Remark 7. Suppose that in Example 6 we have maps

$$l_k \in Hom(V^{\otimes k}, V)_{k-2}^{as} \in B_{-1}(V).$$

If we let $l = l_1 + l_2 + \dots$, then an L_∞ algebra structure on V is given by the symmetric brace relation $l\langle l\rangle = 0$.

4. Some Lemmas

Although the expressions in this paper involve many sums, permutations, and antisymmetrizations, we will be able to simplify things considerably with the help of the following lemmas. Lemma (8) provides a decomposition of $as(f)$ which will be useful later.

Lemma 8. *$as(f) = f \circ \Phi_{nm} \circ \Psi_n \circ \Theta_m \quad \forall\ f \in Hom(V^{\otimes n+m}, V)$, where*

$$\Theta_m(y_1, ..., y_n, z_1, ...z_m) = \sum_{\pi \in S_m} \chi(\pi)(y_1, ..., y_n, z_{\pi(1)}, ...z_{\pi(m)}),$$

$$\Psi_n(y_1, ..., y_n, z_1, ...z_m) = \sum_{\sigma \in S_n} \chi(\sigma)(y_{\sigma(1)}, ..., y_{\sigma(n)}, z_1, ...z_m),$$

$$\Phi_{nm}(y_1, ..., y_n, z_1, ..., z_m) = \sum_{k_0+\cdots+k_n=m} (-1)^{\eta}\left(z_1, ..., z_{k_0}, y_1, z_{1+k_0}, ..., y_n, z_{1+k_0+\cdots+k_{n-1}}, ..., z_m\right),$$

and $\eta = \sum_{i=1}^n \{y_i\left[z_1 + \cdots + z_{(k_0+k_1+\cdots+k_{i-1})}\right] + (n-i)k_i\}$.

Proof. Since Ψ_n does all permutations of the first n inputs, Θ_m provides all permutations of the last m inputs, and Φ_{nm} distributes the last n variables between the first m in every possible way, the composition is clearly a sum of all permutations of the original $n+m$ variables. A moment's reflection also reveals that the sign of each summand in the composition is the Koszul sign together with the sign of the permutation. □

Lemma (9) states that if we sum over all (signed) $(a_1|\dots|a_n)$ unshuffles, and then sum over all (signed) permutations of the a_i variables in each piece, then this is equivalent to just summing over all signed permutations of the original $a_1+\dots+a_n$ variables.

Lemma 9. *If $N = a_1+\dots+a_n$, then $\sum_{\pi\in S_N}\chi(\pi)\big(x_{\pi(1)},\dots,x_{\pi(N)}\big)$ is equal to*

$$\sum_{\substack{\gamma \text{ is}\\ (a_1|\dots|a_n)\\ \text{unshuffle}}}\chi(\gamma)\sum_{\pi_1\in S_{a_1}}\chi(\pi_1)\dots\sum_{\pi_{a_n}\in S_{a_n}}\chi(\pi_n)\Big(x_{\gamma(\pi(1))},\dots,x_{\gamma(\pi_1(a_1))},x_{\gamma(\pi_2(1)+a_1)},\dots,x_{\gamma\left(\pi_n(a_n)+\sum_{i=1}^{n-1}a_i\right)}\Big).$$

Proof. Clearly, the right hand side is the sum of distinct permutations of the x terms with the correct sign. Furthermore, since there are $\frac{N!}{(a_1)!\dots(a_n)!}$ unshuffles γ and $(a_i)!$ permutations π_i, there are $N!$ summands in the right hand side, which agrees with the number of summands on the left hand side. □

Lemma 10. *Suppose $k_0+a_1+k_1+\dots+a_n+k_n = r$, $\sigma\in S_n$, and $\pi\in S_r$. Let $A = a_1+\dots a_n$, denote $X_i = x_{\pi(1+a_1+\dots+a_{i-1})},\dots,x_{\pi(a_1+\dots+a_i)}$, and also denote $X_\pi = x_{\pi(1+A)},\dots,x_{\pi(k_0+A)},X_{\sigma(1)},x_{\pi(1+k_0+A)},\dots,X_{\sigma(n)},x_{\pi(1+k_0+\dots+k_{n-1}+A)},\dots,x_{\pi(r)}$. Then we can define $\hat\pi\in S_r$ by*

$$\hat\pi(i) = \begin{cases} \pi\left(i+A-\sum_{j\le m}a_{\sigma(j)}\right) & \text{if } \sum_{j<m}k_j+\sum_{j\le m}a_{\sigma(j)} < i \le \sum_{j\le m}k_j+\sum_{j\le m}a_{\sigma(j)}.\\ \pi\left(i-\sum_{j<m}k_j+\sum_{j<\sigma(m)}a_j\right) & \text{if } \sum_{j<m}k_j+\sum_{j<m}a_{\sigma(j)} < i \le \sum_{j<m}k_j+\sum_{j\le m}a_{\sigma(j)}.\end{cases}$$

Furthermore, given this notation,

$$X_\pi = x_{\hat\pi(1)},\dots,x_{\hat\pi(r)} \quad \text{and} \quad \epsilon(\hat\pi) = \epsilon(\pi)(-1)^{\alpha_1} \quad \text{and} \quad \chi(\hat\pi) = \chi(\pi)(-1)^{\alpha_2},$$

$$\text{where } \alpha_1 = \sum_{i<j\ \&\ \sigma(i)>\sigma(j)}|X_{\sigma(i)}|\,|X_{\sigma(j)}| + \sum_{i=1}^{n}|X_{\sigma(i)}|\left[x_{\pi(1+A)}+\dots+x_{\pi(k_0+\dots+k_{i-1}+A)}\right]$$

$$\text{and } \alpha_2 = \alpha_1 + \sum_{i<j\ \&\ \sigma(i)>\sigma(j)}a_{\sigma(i)}a_{\sigma(j)} + \sum_{j<i}a_{\sigma(i)}k_j.$$

Proof. Careful examination of the definition of $\hat\pi$ reveals that the first formula moves "free" strings of the form $x_{\pi(1+k_0+\dots+k_{i-1})},\dots,x_{\pi(k_0+\dots+k_i)}$ into place (for $0\le m\le n$), and the second formula relocates the strings $X_{\sigma(i)}$ (for $1\le m\le n$). Thus $X_\pi = x_{\hat\pi(1)},\dots,x_{\hat\pi(r)}$.

Furthermore, when $x_{\pi(1)},\dots,x_{\pi(r)}$ are permuted to yield $x_{\hat\pi(1)},\dots,x_{\hat\pi(r)}$, the Koszul sign is $(-1)^{\alpha_1}$, where the first sum in α_1 comes from σ permuting the X_i strings, and the second sum comes from moving the "free" strings into place.

Finally, the additional sums in α_2 count the transpositions, yielding the correct antisymmetric Koszul sign. □

Lemma 11. *Suppose that $\sigma \in S_n$ permutes $\{v_1 \dots v_n\}$ and $\{w_1 \dots w_n\}$. Then*

(1) $\sum_{i>j} v_i w_j + \sum_{i<j \ \& \ \sigma(i)>\sigma(j)} \{w_{\sigma(i)} v_{\sigma(j)} + v_{\sigma(i)} w_{\sigma(j)}\} + \sum_{i>j} v_{\sigma(i)} w_{\sigma(j)} \equiv 0 \ (mod\ 2).$

(2) $\sum_{i<j \ \& \ \sigma(i)>\sigma(j)} \{v_{\sigma(i)} + v_{\sigma(j)}\} \equiv \sum_i (i-1) v_i + \sum_i (i-1) v_{\sigma(i)} \ (mod\ 2).$

Proof. To prove the first assertion, we note that

$$\sum_{i<j \ \& \ \sigma(i)>\sigma(j)} \{v_{\sigma(i)} w_{\sigma(j)} + w_{\sigma(i)} v_{\sigma(j)}\} + \sum_{i>j} v_{\sigma(i)} w_{\sigma(j)} = \sum_{i<j \& \sigma(i)>\sigma(j)} v_{\sigma(i)} w_{\sigma(j)} + \sum_{i>j \& \sigma(i)<\sigma(j)} v_{\sigma(i)} w_{\sigma(j)} + \sum_{i>j} v_{\sigma(i)} w_{\sigma(j)},$$

which is congruent $(mod\ 2)$ to $\sum_{i<j \& \sigma(i)>\sigma(j)} v_{\sigma(i)} w_{\sigma(j)} + \sum_{i>j \& \sigma(i)>\sigma(j)} v_{\sigma(i)} w_{\sigma(j)} = \sum_{\sigma(i)>\sigma(j)} v_{\sigma(i)} w_{\sigma(j)} = \sum_{i>j} v_i w_j$.

To prove the second statement, suppose that all w_i are odd. Then

$$\sum_{i<j \ \& \ \sigma(i)>\sigma(j)} \{v_{\sigma(i)} + v_{\sigma(j)}\} \equiv \sum_{i<j \ \& \ \sigma(i)>\sigma(j)} \{v_{\sigma(i)} w_{\sigma(j)} + w_{\sigma(i)} v_{\sigma(j)}\} \equiv \sum_{i>j} \{v_i w_j + v_{\sigma(i)} w_{\sigma(j)}\}$$

(by the first assertion). Since all w-terms are odd, this is congruent to

$$\sum_{j=1}^{n} \sum_{i=j+1}^{n} (v_i + v_{\sigma(i)}) = \sum_j (j-1) v_j + \sum_j (j-1) v_{\sigma(j)}.$$

□

5. Symmetrization of Brace Algebras

Given a (non-symmetric) brace structure $\{,\}$ on a graded vector space, we can define a symmetric brace structure $\langle , \rangle$ via

$$f\langle g_1, \dots, g_n \rangle := \sum_{\sigma \in S_n} \epsilon(\sigma) f\{g_{\sigma(1)}, \dots, g_{\sigma(n)}\}.$$

Clearly, this satisfies the first symmetric brace axiom, since $f\langle\ \rangle = f\{\ \} = f$. We show in Theorem (15) that it satisfies the second symmetric brace axiom given in Definition (5), so this does in fact induce a symmetric brace structure. First, however, we need the following two lemmas, which are analogous to Lemmas (8) and (9).

Lemma 12. $\sum_{\rho \in S_{n+m}} \epsilon(\rho) f\{x_{\rho(1)}, \dots, x_{\rho(n)}\} = \tilde{f}_n \circ \theta_m(x_1, \dots, x_{n+m})$, *where*

$\Theta_m(y_1,\dots,y_n,z_1,\dots z_m) = \sum_{\pi \in S_m} \epsilon(\pi)(y_1,\dots,y_n,z_{\pi(1)},\dots z_{\pi(m)})$ *and*

$$\tilde{f}_n(y_1,\dots,y_n,z_1,\dots,z_m) = \sum_{\sigma \in S_n} \epsilon(\sigma) \sum_{k_0+\dots+k_n=m} (-1)^\eta f\{z_1,\dots,z_{k_0},y_{\sigma(1)},z_{1+k_0},\dots,y_{\sigma(n)},z_{1+k_0+\dots+k_{n-1}},\dots,z_m\},$$

with a Koszul sign given by $\eta = \sum_{i=1}^{n} y_{\sigma(i)} \left[z_1 + \dots + z_{(k_0+k_1+\dots+k_{i-1})}\right]$.

Lemma 13. *If* $N = a_1 + \cdots + a_n$*, then* $\sum_{\pi \in S_N} \epsilon(\pi)\big(x_{\pi(1)}, \ldots, x_{\pi(N)}\big)$ *is equal to*

$$\sum_{\substack{\gamma \text{ is} \\ (a_1|\ldots|a_n) \\ \text{unshuffle}}} \epsilon(\gamma) \sum_{\pi_1 \in S_{a_1}} \epsilon(\pi_1) \ldots \sum_{\pi_{a_n} \in S_{a_n}} \epsilon(\pi_n) \Big(x_{\gamma(\pi(1))}, \ldots, x_{\gamma(\pi_1(a_1))}, x_{\gamma(\pi_2(1)+a_1)}, \ldots, x_{\gamma\left(\pi_n(a_n)+\sum_{i=1}^{n-1} a_i\right)}\Big).$$

Remark 14. Although a brace structure allows operators g which accept an arbitrary number of inputs, it will be convenient in the proof of the following theorem to let g^a denote the restriction of g which accepts only exactly a inputs.

Theorem 15. *Given a (non-symmetric) brace structure* $\{\,,\}$ *on a graded vector space, define* $\langle\,,\rangle$ *via*

$$f\langle g_1, \ldots, g_n\rangle := \sum_{\sigma \in S_n} \epsilon(\sigma) f\{g_{\sigma(1)}, \ldots, g_{\sigma(n)}\}.$$

Then $f\langle g_1, \ldots, g_n\rangle\langle x_1, \ldots, x_r\rangle$ *is equal to*

$$\sum_{\substack{\gamma \text{ is } (n+1) \\ \text{unshuffle}}} \epsilon \cdot f\Big\langle g_1\big\langle x_{\gamma(1)}, \ldots, x_{\gamma(a_1)}\big\rangle, \ldots, g_n\big\langle x_{\gamma\left(1+\sum_{i=1}^{n-1} a_i\right)}, \ldots, x_{\gamma\left(\sum_{i=1}^{n} a_i\right)}\big\rangle, x_{\gamma\left(1+\sum_{i=1}^{n} a_i\right)}, \ldots, x_{\gamma(r)}\Big\rangle,$$

where ϵ *is the Koszul sign of the permutation which maps* $(g_1, \ldots, g_n, x_1, \ldots, x_r)$ *to*

$$\Big(g_1, x_{\gamma(1)}, \ldots, x_{\gamma(a_1)}, g_2, \ldots, x_{\gamma\left(1+\sum_{i=1}^{n-1} a_i\right)}, \ldots, x_{\gamma\left(\sum_{i=1}^{n} a_i\right)}, g_n, x_{\gamma\left(1+\sum_{i=1}^{n} a_i\right)}, \ldots, x_{\gamma(r)}\Big).$$

Proof. First, we will look at the right hand side.

If we temporarily denote

$$\begin{aligned} h_k &= g_k\big\langle x_{\gamma(1+a_1+\cdots+a_{k-1})}, \ldots, x_{\gamma(a_1+\cdots+a_k)}\big\rangle \\ &= \sum_{\pi_k \in S_{a_k}} \epsilon(\pi_k)\; g_k\left\{x_{\gamma(\pi_k(1)+a_1+\cdots+a_{k-1})}, \ldots, x_{\gamma(\pi_k(a_k)+a_1+\cdots+a_{k-1})}\right\}, \end{aligned}$$

and denote $A = \sum_{i=1}^{n} a_i$, then the right hand side is equal to

$$\sum_{\substack{a_1+\cdots+a_{n+1}=r \ \& \\ \gamma \text{ is } (a_1|\ldots|a_{n+1}) \text{ unshuffle}}} (-1)^{\nu}\; \epsilon(\gamma)\; f\big\langle h_1, \ldots, h_n, x_{\gamma(1+A)}, \ldots, x_{\gamma(a_{n+1}+A)}\big\rangle,$$

where $\nu = \sum_{i=2}^{n} g_i[x_{\gamma(1)} + \cdots + x_{\gamma(a_1+\cdots+a_{i-1})}]$ is a Koszul sign. After applying Lemma (12), this is equal to

$$\sum_{\substack{a_1+\cdots+a_{n+1}=r, \\ \gamma \text{ is unshuffle}}} (-1)^{\nu}\; \epsilon(\gamma)\; \tilde{f}_n \left(\sum_{\pi_{n+1} \in S_{a_{n+1}}} \epsilon(\pi_{n+1})(h_1, \ldots, h_n, x_{\gamma(\pi_{n+1}(1)+A)}, \ldots, x_{\gamma(\pi_{n+1}(a_n+1)+A)}\right),$$

where $\tilde{f}_n$ is as defined in Lemma (12). Now, we will pull all of the x terms back out, in order to apply Lemma (13). Note that the Koszul signs from this transformation merely cancel out $(-1)^{\nu}$. We then have the following long formula:

$$\sum_{(a_i),\gamma}\epsilon(\gamma)\sum_{\pi_1\in S_{a_1}}\epsilon(\pi_1)\ldots\sum_{\pi_{(n+1)}\in S_{a_{n+1}}}\epsilon(\pi_{n+1})\tilde{f}_n(g_1^{a_1},\ldots,g_n^{a_n},1^{a_{n+1}})\big(x_{\gamma(\pi_1(1))},\ldots,x_{\gamma(\pi_1(a_1))},$$
$$x_{\gamma(\pi_2(1)+a_1)},\ldots,x_{\gamma(\pi_n(A)},$$
$$x_{\gamma(\pi_{n+1}(1)+A)},\ldots x_{\gamma(\pi_{n+1}(a_n+1)+A)}\big).$$

Now, though, we can apply Lemma (13), which yields the much shorter formula,

$$\sum_{(a_i)}\sum_{\pi\in S_r}\epsilon(\pi)\,\tilde{f}_n(g_1^{a_1},\ldots,g_n^{a_n},1^{a_{n+1}})\big(x_{\pi(1)},\ldots,x_{\pi(r)}\big).$$

Before continuing, we need to pull all of the x terms back inside. In order to make our expressions a bit shorter, let X_i denote the input to g_i. In other words, define

$$X_i = x_{\pi(1+a_1+\cdots+a_{i-1})},\ldots,x_{\pi(a_1+\cdots+a_i)} \quad for\ i\in\{1\ldots n\}.$$

It will also be convenient to let $|X_i|$ denote the sum of the degrees of the variables in X_i. When we pull the x-terms inside and use the more concise notation just defined, the formula for the right hand side becomes

$$\sum_{(a_i)}\sum_{\pi\in S_r}\epsilon(\pi)\,(-1)^{\tilde{\nu}}\tilde{f}_n\big(g_1(X_1),\ldots,g_n(X_n),x_{\pi(1+A)},\ldots,x_{\pi(r)}\big),$$

where $\tilde{\nu}=\sum_{j<i}g_i|X_j|$. After expanding $\tilde{f}_n$, the right hand side is equal to

$$\sum_{(a_i)}\sum_{\pi\in S_r}\epsilon(\pi)(-1)^{\tilde{\nu}}\sum_{\sigma\in S_n}\epsilon(\sigma)\sum_{k_0+\cdots+k_n=a_{n+1}}(-1)^{\eta}f\big\{x_{\pi(1+A)},\ldots,x_{\pi(k_0+A)},g_{\sigma(1)}(X_{\sigma(1)}),x_{\pi(1+k_0+A)},\ldots$$
$$\ldots,g_{\sigma(n)}(X_{\sigma(n)}),x_{\pi(1+k_0+\cdots+k_n+A)},\ldots,x_{\pi(r)}\big\}.$$

Here, $\eta=\sum_{i=1}^{n}\big(g_{\sigma(i)}+|X_{\sigma(i)}|\big)\big[x_{\pi(1+A)}+\cdots+x_{\pi(k_0+\cdots+k_{i-1}+A)}\big]$

and $\epsilon(\sigma)=(-1)^{\lambda}$, where $\lambda=\sum_{i<j\ \&\ \sigma(i)>\sigma(j)}\big(g_{\sigma(i)}+|X_{\sigma(i)}|\big)\big(g_{\sigma(j)}+|X_{\sigma(j)}|\big)$.

Now, we will look at the left hand side. $f\big\langle g_1,\ldots,g_n\big\rangle\big\langle x_1,\ldots,x_r\big\rangle$ is equal to $\sum_{\sigma\in S_n}\epsilon(\sigma)f\{g_{\sigma(1)},\ldots,g_{\sigma(n)}\}\big\langle x_1,\ldots,x_r\big\rangle$, which is equal to

$$\sum_{\sigma\in S_n}\epsilon(\sigma)\sum_{\pi\in S_r}\epsilon(\pi)\ f\{g_{\sigma(1)},\ldots,g_{\sigma(n)}\}\{x_{\pi(1)},\ldots,x_{\pi(r)}\}.$$

If we apply Definition (1) and let $g_i^{a_i}$ denote the restriction of g_i which accepts exactly a_i inputs, then the left hand side is equal to

$$\sum_{\sigma\in S_n}\epsilon(\sigma)\sum_{\pi\in S_r}\epsilon(\pi)\sum_{k_0+\cdots+k_n+a_1+\cdots+a_n=r}f\{1^{k_0},g_{\sigma(1)}^{a_{\sigma(1)}},1^{k_1},\ldots,g_{\sigma(n)}^{a_{\sigma(n)}},1^{k_n}\}(x_{\pi(1)},\ldots,x_{\pi(r)}).$$

After applying Lemma (10), this is equal to

$$\sum_{\sigma\in S_n}\epsilon(\sigma)\sum_{(k_i,a_i)}\sum_{\pi\in S_r}\epsilon(\pi)(-1)^{\alpha_1}f\{1^{k_0},g_{\sigma(1)}^{a_{\sigma(1)}},1^{k_1},\ldots,g_{\sigma(n)}^{a_{\sigma(n)}},1^{k_n}\}\big(x_{\pi(1+A)},\ldots,x_{\pi(k_0+A)},X_{\sigma(1)},$$
$$x_{\pi(1+k_0+A)},\ldots,X_{\sigma(n)},$$
$$x_{\pi(1+k_0+\cdots+k_n+A)},\ldots,x_{\pi(r)}\big),$$

where α_1 is given in Lemma (10). Finally, when the x-terms are moved inside, the left hand side is equal to

$$\sum_{\sigma\in S_n}\epsilon(\sigma)\sum_{(k_i,a_i)}\sum_{\pi\in S_r}\epsilon(\pi)(-1)^{\alpha_1+\mu}f\{x_{\pi(1+A)},\dots,x_{\pi(k_0+A)},g_{\sigma(1)}(X_{\sigma(1)}),x_{\pi(1+k_0+A)},\dots$$
$$\dots,g_{\sigma(n)}(X_{\sigma(n)}),x_{\pi(1+k_0+\cdots+k_n+A)},\dots,x_{\pi(r)}\}.$$

Here, $\mu=\sum_i g_{\sigma(i)}[x_{\pi(1+A)}+\cdots+x_{\pi(k_0+\cdots+k_{i-1}+A]}+\sum_{j<i}g_{\sigma(i)}|X_{\sigma(j)}|$

and $\epsilon(\sigma)=(-1)^\zeta$, where $\zeta=\sum_{i<j\ \&\ \sigma(i)>\sigma(j)}g_{\sigma(i)}g_{\sigma(j)}$.

Now that the terms on both sides are easy to compare, it is clear that the two sides are equal if and only if $\tilde{\nu}+\lambda+\eta+\zeta+\alpha_1+\mu\equiv 0\ (mod\ 2)$.

After making the most obvious cancellations, we see that $\tilde{\nu}+\lambda+\eta+\zeta+\alpha_1+\mu$ is congruent to

$$\sum_{j<i}g_i|X_j|+\sum_{i<j\ \&\ \sigma(i)>\sigma(j)}\left(g_{\sigma(i)}|X_{\sigma(j)}|+g_{\sigma(j)}|X_{\sigma(i)}|\right)+\sum_{j<i}g_{\sigma(i)}|X_{\sigma(j)}|,$$

which is congruent to zero (mod 2) by Lemma (11). □

6. Symmetrization of the Brace Structure on $\bigoplus_{k\geq 1}Hom(V^{\otimes k},V)$

In this section, we will demonstrate a nice relationship between the the brace defined in Example 2 and the symmetric brace defined in Example 6, by showing that the symmetrization of the non symmetric brace structure on $Hom(V^{\otimes k},V)$ is equal to the symmetric brace of the anti-symmetrized maps. Specifically, we have

Theorem 16. $\sum_{\sigma\in S_n}\epsilon(\sigma)as(f\{g_{\sigma(1)},\dots,g_{\sigma(n)}\})=as(f)\langle as(g_1),\dots,as(g_n)\rangle$.

Proof. First, we will manipulate the right hand side. Using the symmetric brace structure defined in Example 6, $as(f)\langle as(g_1),\dots,as(g_n)\rangle(x_1,\dots,x_r)$ is equal to

$$(-1)^\delta\sum_{\substack{\gamma\text{ is an}\\(a_1|a_2|\dots|a_{n+1})\\\text{unshuffle}}}\chi(\gamma)as(f)(as(g_1)\otimes\cdots\otimes as(g_n)\otimes 1^{\otimes N-n})(x_{\gamma(1)},\dots,x_{\gamma(r)}),$$

where δ is given in Example 6.

When we substitute the x terms using the Koszul convention and suppress the tensor notation, this is equal to

$$(-1)^\delta\sum_\gamma\chi(\gamma)(-1)^\nu as(f)(h_1,\dots,h_n,x_{\gamma(1+\sum_{i=1}^n a_i)},\dots,x_{\gamma(r)}),$$

where $\nu=\sum_{i=2}^n q_i[x_{\gamma(1)}+\cdots+x_{\gamma(a_1+\cdots+a_{i-1})}]$

and $h_k=as(g_k)\left(x_{\gamma(1+a_1+\cdots+a_{k-1})},\dots,x_{\gamma(a_1+\cdots+a_k)}\right)$

$$=\sum_{\pi_k\in S_{a_k}}\chi(\pi_k)g_k\left(x_{\gamma(\pi_k(1)+a_1+\cdots+a_{k-1})},\dots,x_{\gamma(\pi_k(a_k)+a_1+\cdots+a_{k-1})}\right).$$

If we denote $A = \sum_{i=1}^{n} a_i$ and apply Lemma (8), this is equal to

$$\sum_{\gamma} \chi(\gamma)(-1)^{\delta+\nu} f \circ \Phi_{na} \circ \Psi_n \left(\sum_{\pi_{a_n+1} \in S_{a_{n+1}}} \chi(\pi_{n+1})(h_1, \dots, h_n, x_{\gamma(\pi_{a_n+1}(1)+A)}, \dots, x_{\gamma(\pi_{a_n+1}(a_n+1)+A)} \right).$$

Now, we will pull all of the x terms back out, in order to apply Lemma (9). Note that the Koszul signs from this transformation merely cancel out $(-1)^{\nu}$. We then have the following long formula, which spans two lines!

$$(-1)^{\delta} \sum_{\gamma} \chi(\gamma) \sum_{\pi_1 \in S_{a_1}} \chi(\pi_1) \dots \sum_{\pi_{(a_n+1)} \in S_{a_{n+1}}} \chi(\pi_{n+1}) f \circ \Phi_{na} \circ \Psi_n(g_1, \dots, g_n, 1^{a_{n+1}})$$
$$\left(x_{\gamma(\pi_1(1))}, \dots, x_{\gamma(\pi_1(a_1))}, x_{\gamma(\pi_2(1)+a_1)}, \dots, x_{\gamma(\pi_{a_n}(A)}, x_{\gamma(\pi_{a_n+1}(1)+A)}, \dots, x_{\gamma(\pi_{a_n+1}(a_n+1)+A)} \right).$$

Now, though, we can apply Lemma (9), which yields the much shorter formula,

$$(-1)^{\delta} \sum_{\pi \in S_r} \chi(\pi) f \circ \Phi_{na} \circ \Psi_n(g_1, \dots, g_n, 1^{a_{n+1}}) \left(x_{\pi(1)}, \dots, x_{\pi(r)} \right).$$

Before continuing, we need to pull all of the x terms back inside. In order to make our expressions a bit shorter, let X_i denote the input to g_i, and let X_{n+1} denote the free x terms (letting $a_{n+1} = N - n$). In other words, define

$$X_i = x_{\pi(1+a_1+\cdots+a_{i-1})}, \dots, x_{\pi(a_1+\cdots+a_i)}.$$

It will also be convenient to let $|X_i|$ denote the sum of the degrees of the variables in X_i. When we pull the x-terms inside and use the more concise notation just defined, the formula for the right hand side becomes

$$(-1)^{\delta} \sum_{\pi \in S_r} \chi(\pi)(-1)^{\tilde{\nu}} f \circ \Phi_{n,N-n} \circ \Psi_n \left(g_1(X_1), \dots, g_n(X_n), X_{n+1} \right),$$

where $\tilde{\nu} = \sum_{j<i} q_i |X_j|$. After expanding Ψ_n, the right hand side is equal to

$$(-1)^{\delta} \sum_{\pi \in S_r} \chi(\pi)(-1)^{\tilde{\nu}} f \circ \Phi_{n,N-n} \left(\sum_{\sigma \in S_n} \chi(\sigma) \left(g_{\sigma(1)}(X_{\sigma(1)}), \dots, g_{\sigma(n)}(X_{\sigma(n)}), X_{n+1} \right) \right).$$

In the above expression, $\chi(\sigma)$ is equal to $(-1)^{\lambda}$, where

$$\lambda = \sum_{\substack{i<j\leq n, \\ \sigma(i)>\sigma(j)}} \left[\left(q_{\sigma(i)} + |X_{\sigma(i)}| \right) \left(q_{\sigma(j)} + |X_{\sigma(j)}| \right) + 1 \right].$$

Now, if we expand $\Phi_{n,N-n}$, we get

$$\sum_{\substack{\pi \in S_r, \sigma \in S_n, \\ k_0+\cdots+k_n=N-n}} \chi(\pi)(-1)^{\delta+\tilde{\nu}+\lambda+\eta} f \left(x_{\pi(1+A)}, \dots, x_{\pi(k_0+A)}, g_{\sigma(1)}(X_{\sigma(1)}), x_{\pi(1+k_0+A)}, \dots \right.$$
$$\left. \dots, g_{\sigma(n)}(X_{\sigma(n)}), x_{\pi(1+k_0+\cdots+k_n+A)}, \dots, x_{\pi(r)} \right),$$

where $\eta = \sum_{i=1}^{n} \left\{ \left(q_{\sigma(i)} + |X_{\sigma(i)}| \right) \left(x_{\pi(1+A)} + \cdots + x_{\pi(k_0+\cdots+k_{i-1}+A)} \right) + (n-i)k_i \right\}.$

Now, we will work with the left hand side of the equation. Using the brace defined in Example 2, $\sum_{\sigma\in S_n}\epsilon(\sigma)as(f\{g_{\sigma(1)},\ldots,g_{\sigma(n)}\})(x_1,\ldots,x_r)$ is equal to

$$\sum_{\sigma\in S_n}\varepsilon(\sigma)as\left(\sum_{k_0+\ldots k_n=N-n}(-1)^{\beta}f(1^{\otimes k_0}\otimes g_{\sigma(1)}\otimes 1^{\otimes k_1}\otimes\cdots\otimes 1^{\otimes k_{n-1}}\otimes g_{\sigma(n)}\otimes 1^{\otimes k_n})\right)(x_1,\ldots,x_r),$$

where β is given in Example 2. Note also that the Koszul sign $\epsilon(\sigma)$ must be calculated using the degree of g_i as an element of the symmetric brace algebra (so $|g_i| = q_i + a_i - 1$). Thus $\epsilon(\sigma) = (-1)^{\zeta}$, where

$$\zeta = \sum_{i<j\ \&\ \sigma(i)>\sigma(j)}(q_{\sigma(i)} + a_{\sigma(i)} - 1)(q_{\sigma(j)} + a_{\sigma(j)} - 1).$$

If we now antisymmetrize by taking all signed permutations of the x's, and suppress the tensor notation, this is equal to

$$\sum_{\sigma\in S_n}\sum_{k_0+\ldots k_n=N-n}(-1)^{\beta+\zeta}f\left(1^{k_0},g_{\sigma(1)},1^{k_1},\ldots,1^{k_{n-1}},g_{\sigma(n)},1^{k_n}\right)\left(\sum_{\pi\in S_r}\chi(\pi)\left(x_{\pi(1)},\ldots,x_{\pi(r)}\right)\right).$$

After applying Lemma (10), the left hand side is equal to

$$\sum_{\substack{\pi\in S_r,\sigma\in S_n,\\ k_0+\cdots+k_n=N-n}}(-1)^{\beta+\zeta+\alpha_2}\chi(\pi)f\left(1^{k_0},g_{\sigma(1)},1^{k_1},\ldots,g_{\sigma(n)},1^{k_n}\right)\big(x_{\pi(1+A)},\ldots,x_{\pi(k_0+A)},X_{\sigma(1)},$$
$$x_{\pi(1+k_0+A)},\ldots,X_{\sigma(n)},$$
$$x_{\pi(1+k_0+\cdots+k_n+A)},\ldots,x_{\pi(r)}\big),$$

where α_2 is given in Lemma (10).

Finally, when the variables are moved inside, the left hand side is equal to

$$\sum_{\substack{\pi\in S_r,\sigma\in S_n,\\ k_0+\cdots+k_n=N-n}}(-1)^{\beta+\zeta+\alpha+\mu}\chi(\pi)f\big(x_{\pi(1+A)},\ldots,x_{\pi(k_0+A)},g_{\sigma(1)}(X_{\sigma(1)}),x_{\pi(1+k_0+A)},\ldots$$
$$\ldots,g_{\sigma(n)}(X_{\sigma(n)}),x_{\pi(1+k_0+\cdots+k_n+A)},\ldots,x_{\pi(r)}\big),$$

where $\mu = \sum_i q_{\sigma(i)}[x_{\pi(1+A)} + \cdots + x_{\pi(k_0+\cdots+k_{i-1}+A}] + \sum_{j<i} q_{\sigma(i)}|X_{\sigma(j)}|$.

Since the right hand side is equal to

$$\sum_{\substack{\pi\in S_r,\sigma\in S_n,\\ k_0+\cdots+k_n=N-n}}\chi(\pi)(-1)^{\delta+\tilde{\nu}+\lambda+\eta}f\big(x_{\pi(1+A)},\ldots,x_{\pi(k_0+A)},g_{\sigma(1)}(X_{\sigma(1)}),x_{\pi(1+k_0+A)},\ldots$$
$$\ldots,g_{\sigma(n)}(X_{\sigma(n)}),x_{\pi(1+k_0+\cdots+k_n+A)},\ldots,x_{\pi(r)}\big),$$

we see that the two sides are equal if and only if

$$\beta+\zeta+\alpha_2+\mu+\delta+\tilde{\nu}+\lambda+\eta \equiv 0\ (mod\ 2).$$

After cancelling the most obvious terms, $\beta+\zeta+\alpha_2+\mu+\delta+\tilde{\nu}+\lambda+\eta$ is congruent to

$$\sum_i (n-i)a_{\sigma(i)} + \sum_i (N-i)q_{\sigma(i)} + \sum_{j<i} q_{\sigma(i)}a_{\sigma(j)}$$
$$+\sum_{i<j\ \&\ \sigma(i)>\sigma(j)} [q_{\sigma(i)}a_{\sigma(j)} + q_{\sigma(i)} + a_{\sigma(i)}q_{\sigma(j)} + a_{\sigma(i)} + q_{\sigma(j)} + a_{\sigma(j)}] + \sum_{j<i} q_{\sigma(i)}|X_{\sigma(j)}|$$
$$+\sum_i (N-i)q_i + \sum_{j<i} q_i a_j + \sum_i (n-i)a_i + \sum_{j<i} q_i|X_j| + \sum_{i<j\ \&\ \sigma(i)>\sigma(j)} \left[q_{\sigma(i)}|X_{\sigma(j)}| + |X_{\sigma(i)}|q_{\sigma(j)}\right].$$

After applying Lemma (11), this is congruent to

$$\sum_i \left\{(n-i)a_{\sigma(i)} + (N-i)q_{\sigma(i)} + (i-1)\left[a_i + a_{\sigma(i)} + q_i + q_{\sigma(i)}\right] + (N-i)q_i + (n-i)a_i\right\},$$

which is equal to $\sum_i \left\{(n-1)\left[a_{\sigma(i)} + a_i\right] + (N-1)\left[q_{\sigma(i)} + q_i\right]\right\} \equiv 0\ (mod\ 2)$. □

As a corollary, we obtain Theorem 3.1 of [4]:

Corollary 17. *The anti-symmetrization $l := as(\mu)$ of an A_∞- algebra structure μ yields an L_∞-algebra structure.*

Proof. Given $\mu\{\mu\} = 0$ (recall Remarks (3) and(7)) we have

$$0 = as(\mu\{\mu\}) = as(\mu)\langle as(\mu)\rangle = l\langle l\rangle.$$

□

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Max-Planck-Institute for Gravitational Physics, D-14476 Golm, Germany
E-mail address: `Marilyn.Daily@aei.mpg.de`

Department of Mathematics, North Carolina State University, Raleigh NC 27695
E-mail address: `lada@math.ncsu.edu`