SH-LIE ALGEBRAS INDUCED BY GAUGE TRANSFORMATIONS

RON FULP, TOM LADA, AND JIM STASHEFF

Abstract. Traditionally symmetries of field theories are encoded via Lie group actions, or more generally, as Lie algebra actions. A significant generalization is required when 'gauge parameters' act in a field dependent way. Such symmetries appear in several field theories, most notably in a 'Poisson induced' class due to Schaller and Strobl [SS94] and to Ikeda[Ike94], and employed by Cattaneo and Felder [CF99] to implement Kontsevich's deformation quantization [Kon97]. Consideration of 'particles of spin > 2 led Berends, Burgers and van Dam [Bur85, BBvD84, BBvD85] to study 'field dependent parameters' in a setting permitting an analysis in terms of smooth functions. Having recognized the resulting structure as that of an sh-lie algebra (L_∞-algebra), we have now formulated such structures entirely algebraically and applied it to a more general class of theories with field dependent symmetries.

1. Introduction

Ever since the discovery of Yang-Mills theory, physicists have been intrigued by the different manifestations of symmetries in field theories. Symmetries in gravitational theories are induced by spacetime transformations which preserve the spacetime structure whereas Yang-Mills symmetries are defined via transformations of some internal vector space. Many authors have attempted to reformulate gravitational symmetries in a manner which is compatible with the Yang-Mills approach as quantization of Yang-Mills theories is better understood than most attempts to quantize gravity.

The present paper has as its purpose to show that gauge symmetries of certain field theories have an unexpectedly rich algebraic structure. Traditional theories lead one to expect that the symmetries of field theories are encoded via Lie group actions, or more generally, as Lie algebra actions. We find that the gauge symmetries of many field theories in fact do not arise from a Lie algebra action, but rather from an sh-Lie (or L_∞) algebra action.

Stasheff's research supported in part by the NSF throughout most of his career, most recently under grant DMS-9803435.

The physics of "particles of spin ≤ 2" leads to representations of a Lie algebra Ξ of gauge parameters on a vector space Φ of fields. A significant generalization occurs when the gauge parameters act in a *field dependent* way. By a field dependent action of Ξ on Φ, Berends, Burgers and van Dam [Bur85, BBvD86, BBvD85] mean a polynomial (or power series) map $\delta(\xi)(\phi) = \Sigma_{i\geq 0} T_i(\xi, \phi)$ where T_i is linear in ξ and polynomial of homogeneous degree i in ϕ.

Field dependent gauge symmetries appear in several field theories, most notably in a 'Poisson induced' class due to Schaller and Strobl [SS94] and to Ikeda [Ike94], and employed by Cattaneo and Felder [CF99] to implement Kontsevich's deformation quantization [Kon97]. Ikeda [Ike94] considers two-dimensional and three-dimensional [Ike01] theories with a generalized Yang-Mills field which has values in a so-called nonlinear Lie algebra. He finds that if the non-linear Lie structure is chosen appropriately and if he allows the Yang-Mills field to interact with certain scalar fields, then he can recapture gravitational theories in two dimensions. In this way, two-dimensional gravity is formulated as a Yang-Mills theory and its symmetries arise in the same way as traditional Yang-Mills symmetries. The three-dimensional case [Ike01] provides deformations of physicists' BF theories and analogous results hold in higher dimensions.

Although expressed rather differently, the Berends, Burgers and van Dam approach provides further insight into the algebraic structure of the gauge symmetries of the above class of field theories. In fact their context is more general than that of Ikeda and that of Cattaneo and Felder, since Berends, Burgers and van Dam consider arbitrary field theories, subject only to the requirement that the commutator of two gauge symmetries be another gauge symmetry whose gauge parameter is possibly field dependent. We refer to this requirement as the BBvD hypothesis. Notice Berends, Burgers and van Dam do not require an a priori given Lie structure to induce the algebraic structure of the gauge symmetry "algebra". On the other hand, Ikeda requires a structure called a nonlinear Lie algebra which he uses to obtain symmetries which in turn are used to find a Lagrangian for which the symmetries are gauge symmetries. In this sense, his nonlinear Lie structure drives the entire theory. Similarly, Cattaneo and Felder have a Poisson structure which explicitly appears in both the action of their theory and in their gauge symmetries.

The present work has as its goal to clarify the algebraic structure of the more general gauge "algebra" outlined in Berends, Burgers and van Dam . When the BBvD hypothesis is satisfied, we show that the gauge symmetry algebra of a large class of field theories is an sh-Lie algebra. Of course, as we show, this sh-Lie structure, in special cases, will reduce to the more familiar Lie structures one encounters in various field theories. On the other hand, some of these field theories satisfy the BBvD hypothesis only 'on-shell'. When closure on the original space of parameters is lost, physicists speak of an 'open algebra'. This leads us, in section 7, to a 'generalized BBvD hypothesis' which in turn will allow us to show how the sh-Lie structure must be modified to handle 'off-shell' gauge symmetries.

We formulate the relevant structures in BBvD's theory in terms of linear maps from a certain coalgebra $\Lambda^*\Phi$ into the respective vector spaces Φ of fields and Ξ of gauge parameters. The coalgebra and the algebra structures of $\Lambda^*\Phi$ as well as the Lie algebra structure of $\mathrm{Hom}(\Lambda^*\Phi, \Phi)$ are described in Section 2. It turns out that the space Ξ of gauge parameters has, in general, no natural Lie structure, but the space of linear maps from $\Lambda^*\Phi$ into Ξ is a Lie algebra under certain mild assumptions along with the BBvD hypothesis. This is proved in Section 3. Section 4 provides the reader with a short description of two equivalent methods for defining sh-Lie algebras. Our main result is found in Section 5 where we show that, under the same assumptions required in Section 3, the fields and gauge parameters combine to form an sh-Lie algebra. In Section 6 we show how our results relate to the classical situation in which the space Ξ of gauge parameters is a Lie algebra which acts on the space Φ of fields. Section 7 provides further links to the physics literature where certain sigma-models are known to satisfy the BBvD hypothesis only 'on-shell'. This requires us to further generalize the BBvD hypothesis; consequently these gauge algebras are "on shell" sh-Lie algebras which are not "on shell" Lie algebras. Finally, in Section 8 we show explicitly how our formalism applies to the work of Ikeda [Ike94] on two-dimensional gravitational theories and his study of non-linear Lie algebras. In addition, we show that Ikeda's bracket is the 'non-linear' analog of the Kirillov-Kostant bracket.

We are grateful to Berends, Burgers and van Dam for the inspiration of Burgers' dissertation and especially to van Dam for several discussions as our research developed.

2. Our framework

We work with vector spaces over a field k of characteristic 0 or, more generally, over a commutative k-algebra $\mathcal{A}$, typically, $C^\infty(M)$ for some smooth manifold M. Unless otherwise specified, Hom will denote the $\mathcal{A}$-module of $\mathcal{A}$-linear maps.

Let Φ be a free $\mathcal{A}$-module and let $\Lambda^*\Phi$ denote the free nilpotent graded cocommutative coalgebra over $\mathcal{A}$ cogenerated by Φ with comultiplication denoted Δ. This is the coalgebra of graded symmetric tensors in the full tensor coalgebra on Φ. The $\mathcal{A}$-module Coder$(\Lambda^*\Phi)$ of *coderivations* (over $\mathcal{A}$) on $\Lambda^*\Phi$ is a Lie algebra with bracket given by the commutator with respect to composition. Recall that a coderivation is a linear map $\Theta : \Lambda^*\Phi \to \Lambda^*\Phi$ that satisfies the equation

$$\Delta \circ \Theta = (\Theta \otimes 1 + 1 \otimes \Theta) \circ \Delta.$$

(In the graded situation, the usual Koszul sign conventions are in effect.)

The $\mathcal{A}$-module Hom$(\Lambda^*\Phi, \Phi)$ is isomorphic to Coder$(\Lambda^*\Phi)$ and hence inherits a Lie algebra structure; the bracket on Hom$(\Lambda^*\Phi, \Phi)$ is known as the Gerstenhaber bracket [Ger62, Sta93]. The isomorphism

$$\mathrm{Hom}(\Lambda^*\Phi, \Phi) \ni h \rightleftharpoons \bar{h} \in \mathrm{Coder}(\Lambda^*\Phi)$$

is given by the correspondence

$$\bar{h}(\phi_1 \wedge \cdots \wedge \phi_n) = \sum_{\{unshuff\}} h(\phi_{\sigma(1)} \wedge \cdots \wedge \phi_{\sigma(p)}) \wedge \phi_{\sigma(p+1)} \wedge \cdots \wedge \phi_{\sigma(n)}$$

for $h \in \mathrm{Hom}(\Lambda^p(\Phi), \Phi)$. The set $\{unshuff\}$ is the set of $(p, n-p)$-*unshuffles*, that is, the permutations of $\{1, \ldots, n\}$ such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(n)$. We may write $\bar{h}$ as the composition $\bar{h} = m \circ (h \otimes 1) \circ \Delta$ where m is the usual product in $\Lambda^*\Phi$ regarded as an algebra (symmetric on even elements and skew on odd ones; no compatability with the coproduct is assumed nor needed).

The Gerstenhaber bracket on $Hom(\Lambda^*\Phi, \Phi)$ may be described as $[f, g] = f \circ \bar{g} - g \circ \bar{f}$ where $\bar{f}$ and $\bar{g}$ are the coderivations corresponding to f and g. In this notation the "Gerstenhaber comp" operation may be defined by $f \odot g = f \circ \bar{g}$, for $f, g \in Hom(\Lambda^*\Phi, \Phi)$. Thus an alternative notation for the Lie bracket on $Hom(\Lambda^*\Phi, \Phi)$ is $[f, g] = f \odot g - g \odot f$.

3. A preliminary result

Now let Ξ and Φ be arbitrary $\mathcal{A}$-modules. In the Yang Mills example, the map δ takes gauge parameters to covariant derivatives. In generalizing that, we suppose that we are given a k-linear map $\delta : \Xi \to \mathrm{Hom}(\Lambda^*\Phi, \Phi)$. Formally, we can write $\delta(\xi) = \Sigma_{i=0} T_i(\xi)$ where T_i is 0 except on $\Lambda^i\Phi$. (This T_i is equivalent to the T_i of Berends, Burgers and van Dam .) We extend δ to a map

$$\hat{\delta} : \mathrm{Hom}_k(\Lambda^*\Phi, \Xi) \to \mathrm{Hom}(\Lambda^*\Phi, \Phi)$$

by

$$\hat{\delta}(\pi) = ev \circ (\delta \circ \pi \otimes 1) \circ \Delta$$

where ev is the evaluation map. That is,

$$\hat{\delta}(\pi)(\phi_1\wedge\cdots\wedge\phi_n) = \sum_{\{unshuff\}} \delta(\pi(\phi_{\sigma(1)}\wedge\cdots\wedge\phi_{\sigma(p)})(\phi_{\sigma(p+1)}\wedge\cdots\wedge\phi_{\sigma(n)}).$$

We may think of Ξ as being contained in $\mathrm{Hom}_k(\Lambda^*\Phi,\Xi)$ by identifying $\xi\in\Xi$ with the map, also denoted ξ, in $\mathrm{Hom}_k(\Lambda^*\Phi,\Xi)$ which is 0 except on the scalars where $\xi(1)=\xi$. Note that $\wedge^*\Phi$ is an $\mathcal{A}$ module and $k\subset\mathcal{A}$ and so $1\in k\subset\mathcal{A}$. We will be careful to distinguish k-linear maps from $\mathcal{A}$-linear as the need occurs. It is easy to see that $\hat{\delta}(\xi)=\delta(\xi)$.

Our problem concerns possible algebraic structure on Ξ; consequently we consider the possibility of constructing a Lie-type bracket on $\mathrm{Hom}_k(\Lambda^*\Phi,\Xi)$ via the mapping $\hat{\delta}$. Under certain conditions, such a bracket may then be used to obtain a bracket on the parameter space defined by restricting the induced bracket on $\mathrm{Hom}_k(\Lambda^*\Phi,\Xi)$ to the parameter space Ξ. With this in mind, define

$$[\pi_1,\pi_2]:=\pi_1\circ\overline{\hat{\delta}(\pi_2)}-\pi_2\circ\overline{\hat{\delta}(\pi_1)},$$

for $\pi_1,\pi_2\in\mathrm{Hom}_k(\Lambda^*\Phi,\Xi)$. It turns out that this bracket does not generally satisfy the Jacobi identity. Moreover, if we choose $\pi_1=\xi,\pi_2=\eta\in\Xi$, then

$$[\xi,\eta]=\xi\odot\hat{\delta}(\eta)-\eta\odot\hat{\delta}(\xi)=0,$$

and as a result, the restriction of the induced bracket to Ξ yields an abelian Lie algebra structure. In many cases of interest, the parameter space has an a priori nonabelian Lie algebra structure on it and we would certainly want the Lie structure on $\mathrm{Hom}_k(\Lambda^*\Phi,\Xi)$ to reproduce this structure when restricted to the parameter space Ξ.

In order to assure the Jacobi property of bracket on $\mathrm{Hom}_k(\Lambda^*\Phi,\Xi)$, we introduce a *correction term.* We accomplish this, following Berends, Burgers and van Dam, by assuming that there is a map

$$C:\Xi\otimes\Xi\to\mathrm{Hom}_k(\Lambda^*\Phi,\Xi)$$

such that

$$[\delta(\xi),\delta(\eta)]=\hat{\delta}C(\xi,\eta)\in\mathrm{Hom}(\Lambda^*\Phi,\Phi)$$

for all $\xi,\eta\in\Xi$. We will refer to this as the *BBvD hypothesis.*

Extend C to a mapping

$$\hat{C}:\mathrm{Hom}_k(\Lambda^*\Phi,\Xi)\otimes\mathrm{Hom}_k(\Lambda^*\Phi,\Xi)\to\mathrm{Hom}_k(\Lambda^*\Phi,\Xi)$$

by

$$\hat{C}(\pi_1,\pi_2)=C\circ((\pi_1\otimes\pi_2)\otimes 1)\circ(\Delta\otimes 1)\circ\Delta,$$

where we have identified C with its adjoint mapping, which is the mapping from $\Xi \otimes \Xi \otimes \Lambda^*\Phi$ into Ξ defined by

$$(\xi, \eta, \phi_1 \wedge \cdots \wedge \phi_n) \longrightarrow C(\xi, \eta)(\phi_1 \wedge \cdots \wedge \phi_n).$$

Next, we redefine the bracket on $\mathrm{Hom}_k(\Lambda^*\Phi, \Xi)$ given above by including the correction term C :

$$[\pi_1, \pi_2] := \pi_1 \odot \hat{\delta}(\pi_2) - \pi_2 \odot \hat{\delta}(\pi_1) + \hat{C}(\pi_1, \pi_2).$$

Theorem 1. *The mapping $\hat{\delta}$ preserves brackets; that is, $\hat{\delta}[\pi_1, \pi_2] = [\hat{\delta}(\pi_1), \hat{\delta}(\pi_2)]$. Moreover, if $\hat{\delta} : \mathrm{Hom}_k(\Lambda^*\Phi, \Xi) \to \mathrm{Hom}(\Lambda^*\Phi, \Phi)$ is injective, then $[\pi_1, \pi_2]$ satisfies the Jacobi identity.*

Proof. Observe that if $\pi_1, \pi_2 \in \mathrm{Hom}_k(\Lambda^*\Phi, \Xi)$,

$$\begin{aligned}
\hat{\delta}(\pi_1) \odot \hat{\delta}(\pi_2) =& \hat{\delta}(\pi_1) \circ \overline{\hat{\delta}(\pi_2)} \\
&= ev \circ [(\hat{\delta} \circ \pi_1) \otimes 1] \circ \Delta \circ \overline{\hat{\delta}(\pi_2)} \\
&= ev \circ [(\hat{\delta} \circ \pi_1) \otimes 1] \circ \{(\overline{\hat{\delta}(\pi_2)} \otimes 1) + (1 \otimes \overline{\hat{\delta}(\pi_2)}) \circ \Delta\} \\
&= ev \circ [((\hat{\delta} \circ \pi_1) \circ \overline{\hat{\delta}(\pi_2)}) \otimes 1] \circ \Delta + ev \circ [((\hat{\delta} \circ \pi_1) \otimes \overline{\hat{\delta}(\pi_2)})] \circ \Delta \\
&= \hat{\delta}(\pi_1 \circ \overline{\hat{\delta}(\pi_2)})) + ev \circ [((\hat{\delta} \circ \pi_1) \otimes \overline{\hat{\delta}(\pi_2)})] \circ \Delta \\
&= \hat{\delta}(\pi_1 \odot \hat{\delta}(\pi_2))) + ev \circ [((\hat{\delta} \circ \pi_1) \otimes \overline{\hat{\delta}(\pi_2)})] \circ \Delta.
\end{aligned}$$

It follows that

$$[\hat{\delta}(\pi_1), \hat{\delta}(\pi_2)] = \hat{\delta}(\pi_1 \odot \hat{\delta}(\pi_2))) - \hat{\delta}(\pi_2 \odot \hat{\delta}(\pi_1))) + E$$

where

$$E = ev \circ [((\hat{\delta} \circ \pi_1) \otimes \overline{\hat{\delta}(\pi_2)})] \circ \Delta - ev \circ [(\hat{\delta} \circ \pi_2) \otimes \overline{\hat{\delta}(\pi_1)}] \circ \Delta.$$

This says that E measures the deviation of $\hat{\delta}$ from being a $\mathrm{Coder}(\Lambda^*\Phi)$-module map.

We must show that E is in the image of $\hat{\delta}$. Recall that for $f \in \mathrm{Hom}(\Lambda^*\Phi, \Phi)$, we have $\overline{f} = m \circ (f \otimes 1) \circ \Delta$, where m denotes the algebra (wedge) product on $\Lambda^*\Phi$. Thus

$$\begin{aligned}
(\hat{\delta} \circ \pi_1) \otimes \overline{\hat{\delta}(\pi_2)} &= (\hat{\delta} \circ \pi_1) \otimes \{m \circ (\hat{\delta}(\pi_2) \otimes 1) \circ \Delta\} \\
&= (\hat{\delta} \circ \pi_1) \otimes \{m \circ ([ev \circ ((\hat{\delta} \circ \pi_2) \otimes 1)] \otimes 1) \circ (\Delta \otimes 1) \circ \Delta\}. \\
&= (\hat{\delta} \circ \pi_1) \otimes \{m \circ ([ev \circ ((\hat{\delta} \circ \pi_2) \otimes 1)] \otimes 1) \circ (1 \otimes \Delta) \circ \Delta\}.
\end{aligned}$$

For $F \in \Lambda^*\Phi$, write $\Delta(F) = \sum(F_1 \otimes F_2), \Delta(F_2) = \sum(F_{21} \otimes F_{22})$ and $\Delta(F_{22}) = \sum(F_{221} \otimes F_{222})$. In order to simplify notation we drop the summation symbol wherever the latter coproducts appear below. From our last calculation we have

$$(ev \circ [(\hat{\delta} \circ \pi_1) \otimes \overline{\hat{\delta}(\pi_2)}] \circ \Delta)(F) = [\hat{\delta}(\pi_1(F_1)) \odot \hat{\delta}(\pi_2(F_{21}))](F_{22}),$$

and

$$(ev \circ [(\hat{\delta} \circ \pi_2) \otimes \overline{\Delta(\pi_1)}] \circ \Delta)(F) = [\hat{\delta}(\pi_2(F_1)) \odot \hat{\delta}(\pi_1(F_{21}))](F_{22}).$$

Because Δ is cocommutative, the full summations are equal:

$$\Sigma F_1 \otimes F_{21} \otimes F_{22} = \Sigma F_{21} \otimes F_1 \otimes F_{22}.$$

Thus

$$\begin{aligned} E(F) &= [\hat{\delta}(\pi_1(F_1)), \hat{\delta}(\pi_2(F_{21}))](F_{22}) = \hat{\delta}(C(\pi_1(F_1), \pi_2(F_{21}), F_{221}))(F_{222}) \\ &= ev \circ (\{\hat{\delta} \circ C \circ [(\pi_1 \otimes \pi_2) \otimes 1]\} \otimes 1)(F_1 \otimes F_{21} \otimes F_{221} \otimes F_{222}) \\ &= ev \circ (\{\hat{\delta} \circ C \circ [(\pi_1 \otimes \pi_2) \otimes 1]\} \otimes 1)(F_1 \otimes F_{21} \otimes (\Delta F_{22})) \\ &= ev \circ (\{\hat{\delta} \circ C \circ [(\pi_1 \otimes \pi_2) \otimes 1]\} \otimes 1)(([1 \otimes ((1 \otimes \Delta) \circ \Delta)] \circ \Delta)(F)) \\ &= ev \circ (\{\hat{\delta} \circ C \circ [(\pi_1 \otimes \pi_2) \otimes 1]\} \otimes 1)((1 \otimes 1 \otimes \Delta) \circ (1 \otimes \Delta) \circ \Delta)(F)). \end{aligned}$$

It follows from coassociativity that

$$\begin{aligned} E &= ev \circ (\{\hat{\delta} \circ C \circ [(\pi_1 \otimes \pi_2) \otimes 1]\} \otimes 1) \circ ((\Delta \otimes 1 \otimes 1) \circ (\Delta \otimes 1) \circ \Delta \\ &= ev \circ (\{\hat{\delta} \circ C \circ [(\pi_1 \otimes \pi_2) \otimes 1]\} \otimes 1) \circ ([(\Delta \otimes 1) \circ \Delta] \otimes 1) \circ \Delta \\ &= ev \circ (\{\hat{\delta} \circ C \circ [(\pi_1 \otimes \pi_2) \otimes 1] \circ (\Delta \otimes 1) \circ \Delta\} \otimes 1) \circ \Delta \\ &= ev \circ (\{\hat{\delta} \circ \hat{C}(\pi_1, \pi_2)\} \otimes 1) \circ \Delta = \hat{\delta}(\hat{C}(\pi_1, \pi_2)). \end{aligned}$$

Thus E is in the image of $\hat{\delta}$ and in fact

$$[\hat{\delta}(\pi_1), \hat{\delta}(\pi_2)] = \hat{\delta}(\pi_1 \odot \hat{\delta}(\pi_2) - \pi_2 \odot \hat{\delta}(\pi_1)) + \hat{\delta}(\hat{C}(\pi_1, \pi_2)) = \hat{\delta}([\pi_1, \pi_2]).$$

To verify the Jacobi identity, apply $\hat{\delta}$ to the Jacobi expression in $\mathrm{Hom}_k(\Lambda^*\Phi, \Xi)$. By the morphism condition just established, the result is the Jacobi identity valid in $\mathrm{Hom}(\Lambda^*\Phi, \Phi)$. Assuming that $\hat{\delta}$ is injective, the Jacobi identity in $\mathrm{Hom}_k(\Lambda^*\Phi, \Xi)$ follows. □

This result suggests that the parameter space should be enlarged to include all of $Hom_k(\Lambda^*\Phi, \Xi)$. It turns out that the polynomial equations of physical relevance define an sh-Lie structure on an appropriate graded vector space $\mathbb{L}$. We consider the sh-Lie formalism briefly in the next section.

4. Sh-Lie algebras

We now review the relationship between sh-Lie algebras (L_∞-algebras) and cocommutative coalgebras [LS93, LM95]. Let $(\mathbb{L}, d)$ be a differential graded vector space. If $(\mathbb{L}, d)$ is a chain complex (degree $d = -1$), then an sh-Lie structure on $\mathbb{L}$ is a collection of skew symmetric linear maps $l_n : \mathbb{L}^{\otimes n} \longrightarrow \mathbb{L}$ of degree $n-2$ that satisfy the relations

$$\sum_{i+j=n+1} \sum_{\sigma} e(\sigma)(-1)^{\sigma}(-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), \ldots, x_{\sigma(n)}) = 0$$

where $(-1)^\sigma$ is the sign of the permutation σ, $e(\sigma)$ is the sign that arises from the degrees of the permuted elements and σ is taken over all $(i, n-i)$ unshuffles.

If $l_n = 0$ for $n \geq 3$, this is just the description of a dg Lie algebra.

If $(\mathbb{L}, d)$ is a cochain complex (degree $d = +1$), then the sh-Lie structure on $\mathbb{L}$ is given by skew symmetric linear maps $l_n : \mathbb{L}^{\otimes n} \longrightarrow \mathbb{L}$ of degree $2-n$ that satisfy the same relations.

Let $\uparrow \mathbb{L}$ denote the suspension of the graded vector space $\mathbb{L}$; i.e. $\uparrow \mathbb{L}$ is the graded vector space with $(\uparrow \mathbb{L})_n = \mathbb{L}_{n-1}$; similarly, let $\downarrow \mathbb{L}$ denote the desuspension of $\mathbb{L}$; i.e. $(\downarrow \mathbb{L})_n = \mathbb{L}_{n+1}$.

One may then describe an sh-Lie structure on the chain complex $(\mathbb{L}, d)$ by a coderivation $\overline{D}$ of degree -1 on the coalgebra $\Lambda^*(\uparrow \mathbb{L})$ such that $\overline{D}^2 = 0$; similarly, an sh-Lie structure on the cochain complex $(\mathbb{L}, d)$ is a coderivation $\overline{D}$ of degree $+1$ on the coalgebra $\Lambda^*(\downarrow \mathbb{L})$ such that $\overline{D}^2 = 0$. Equivalently, the sh-Lie structure may be described by a linear mapping $D : \Lambda^*(\downarrow \mathbb{L}) \longrightarrow (\downarrow \mathbb{L})$ such that $D \circ \overline{D} = 0$. The proof of the assertion for chain complexes may be found in [LS93] and [Sta93]; a proof for cochain complexes can be formulated by a straightforward modification of the proof for chain complexes.

5. The gauge algebra is an sh-Lie algebra

We now restrict our attention to the constant maps in $\mathrm{Hom}_k(\Lambda^*\Phi, \Xi)$ and show that our algebraic structure on $\mathrm{Hom}_k(\Lambda^*\Phi, \Xi)$ induces an sh-Lie structure on the graded space $\mathbb{L} = \{\Xi, \Phi\}$. Throughout this section, we assume the BBvD hypothesis and that $\hat{\delta}$ is injective, so Theorem 1 holds and consequently the bracket on $\mathrm{Hom}_k(\Lambda^*\Phi, \Xi)$ defined by

$$[\pi_1, \pi_2] := \pi_1 \odot \hat{\delta}(\pi_2) - \pi_2 \odot \hat{\delta}(\pi_1) + \hat{C}(\pi_1, \pi_2)$$

satisfies the Jacobi identity. By definition,

$$[\delta(\xi), \delta(\eta)] = \delta(\xi) \odot \delta(\eta) - \delta(\eta) \odot \delta(\xi)$$

while the definition of C gives

$$[\delta(\xi), \delta(\eta)] = \hat{\delta}C(\xi, \eta) \in \mathrm{Hom}(\Lambda^*\Phi, \Phi),$$

so our *commutator relation* is

$$\delta(\xi) \odot \delta(\eta) - \delta(\eta) \odot \delta(\xi) = \hat{\delta}(C(\xi, \eta)).$$

The definition of the bracket in $\mathrm{Hom}_k(\Lambda^*\Phi, \Xi)$ restricted to constant maps takes on the form $[\xi_1, , \xi_2] = C(\xi_1, \xi_2)$. Consequently, the Jacobi identity takes on the form

$$[C(\xi_1, \xi_2), \xi_3] - [C(\xi_1, \xi_3), \xi_2] + [C(\xi_2, \xi_3), \xi_1] = 0.$$

Let us examine the first term:

$$[C(\xi_1, \xi_2), \xi_3] = C(\xi_1, \xi_2) \odot \delta(\xi_3) - \xi_3 \odot \hat{\delta}C(\xi_1, \xi_2) + \hat{C}(C(\xi_1, \xi_2), \xi_3) = C(\xi_1, \xi_2) \odot \delta(\xi_3) + \hat{C}(C(\xi_1, \xi_2), \xi_3)$$

because $\xi_3 \odot \hat{\delta}C(\xi_1, \xi_2) = 0$ as ξ is a constant map (non-zero only on scalars). We now add together the results from the remaining two terms and write the *Jacobi relation* as

$$C(\xi_1, \xi_2) \odot \delta(\xi_3) - C(\xi_1, \xi_3) \odot \delta(\xi_2) + C(\xi_2, \xi_3) \odot \delta(\xi_1)$$
$$+\hat{C}(C(\xi_1, \xi_2), \xi_3) - \hat{C}(C(\xi_1, \xi_3), \xi_2) + \hat{C}(C(\xi_2, \xi_3), \xi_1) = 0.$$

For the sh-Lie structure, we first combine the fields and gauge parameters to form a single differential graded vector space $\mathbb{L}$.

Definition 1. *The underlying dg vector space* $\mathbb{L}$ *of* the sh-Lie algebra *has* Ξ *in degree 0,* Φ *in degree 1 and 0 in all other degrees. The differential* $\partial : \Xi \to \Phi$ *is given by* $\partial(\xi) = \delta(\xi)(1) \in \Phi$.

Theorem 2. *The linear map*

$$D : \Lambda^*(\downarrow \mathbb{L}) \to \downarrow \mathbb{L}$$

given by

$$\begin{aligned} D(\xi) =& \partial(\xi) \\ D(\xi \wedge \phi_1 \wedge \cdots \wedge \phi_n) =& \delta(\xi)(\phi_1 \wedge \cdots \wedge \phi_n) \text{ for } n \geq 1 \\ D(\xi_1 \wedge \xi_2 \wedge \phi_1 \wedge \cdots \wedge \phi_n) =& C(\xi_1, \xi_2)(\phi_1 \wedge \cdots \wedge \phi_n) \end{aligned}$$

and $D = 0$ *on elements of* $\Lambda^*(\downarrow \mathbb{L})$ *with more than two entries from* Ξ *or with no entry from* Ξ *gives* $\mathbb{L}$ *the structure of an sh-Lie algebra.*

Remark: Recall that we have assumed as hypothesis for this theorem that the bracket on $\mathrm{Hom}_k(\Lambda^*\Phi, \Xi)$ satisfies the Jacobi identity. According to Theorem 1 this is true if $\hat{\delta}$ is injective. It is not difficult to prove that $\hat{\delta}$ is injective whenever δ is injective. If we replace the original parameter space with the new parameter space $\Xi/ker(\delta)$, one has the sh-Lie structure obtained in the proof below.

Proof. We need only evaluate $D \circ \bar{D}$ on elements of the form $(\xi_1 \wedge \xi_2 \wedge \phi_1 \wedge \cdots \wedge \phi_n)$ and $(\xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \phi_1 \wedge \cdots \wedge \phi_n)$.

We begin with

$$D \circ \bar{D}(\xi_1 \wedge \xi_2 \wedge \phi_1 \wedge \cdots \wedge \phi_n) =$$

$$D\{\sum_\sigma \delta(\xi_1)(\phi_{\sigma(1)} \wedge \cdots \wedge \phi_{\sigma(i)}) \wedge \xi_2 \wedge \phi_{\sigma(i+1)} \wedge \cdots \wedge \phi_{\sigma(n)}$$

$$-\sum_\tau \delta(\xi_2)(\phi_{\tau(1)} \wedge \cdots \wedge \phi_{\tau(j)}) \wedge \xi_1 \wedge \phi_{\tau(j+1)} \wedge \cdots \wedge \phi_{\tau(n)}$$

$$\sum_\rho C(\xi_1, \xi_2)(\phi_{\rho(1)} \wedge \cdots \wedge \phi_{\rho(k)}) \wedge \phi_{\rho(k+1)} \wedge \cdots \wedge \phi_{\rho(n)}\}$$

where σ, τ and ρ are the evident unshuffles.

This composition is equal to

$$D\{\sum_\sigma \xi_2 \wedge \delta(\xi_1)(\phi_{\sigma(1)} \wedge \cdots \wedge \phi_{\sigma(i)}) \wedge \phi_{\sigma(i+1)} \wedge \cdots \wedge \phi_{\sigma(n)}$$

$$-\sum_\tau \xi_1 \wedge \delta(\xi_2)(\phi_{\tau(1)} \wedge \cdots \wedge \phi_{\tau(j)}) \wedge \phi_{\tau(j+1)} \wedge \cdots \wedge \phi_{\tau(n)}$$

$$+\sum_\rho C(\xi_1, \xi_2)(\phi_{\rho(1)} \wedge \cdots \wedge \phi_{\rho(k)}) \wedge \phi_{\rho(k+1)} \wedge \cdots \wedge \phi_{\rho(n)}\}$$

$$= \sum_\sigma \delta(\xi_2)(\delta(\xi_1)(\phi_{\sigma(1)} \wedge \cdots \wedge \phi_{\sigma(i)}) \wedge \phi_{\sigma(i+1)} \wedge \cdots \wedge \phi_{\sigma(n)})$$

$$-\sum_\tau \delta(\xi_1)(\delta(\xi_2)(\phi_{\tau(1)} \wedge \cdots \wedge \phi_{\tau(j)}) \wedge \phi_{\tau(j+1)} \wedge \cdots \wedge \phi_{\tau(n)})$$

$$+\sum_\rho \delta(C(\xi_1, \xi_2))(\phi_{\rho(1)} \wedge \cdots \wedge \phi_{\rho(k)})(\phi_{\rho(k+1)} \wedge \cdots \wedge \phi_{\rho(n)})$$

which is equal to 0 by the commutator relation.

For the terms of the form $(\xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \phi_1 \wedge \cdots \wedge \phi_n)$, the only unshuffles that we need to consider are those that result in terms of the form

$$(\xi_i \wedge \phi_{\sigma(1)} \wedge \cdots \wedge \phi_{\sigma(p)} \wedge \xi_j \wedge \xi_k \wedge \phi_{\sigma(p+1)} \wedge \cdots \wedge \phi_{\sigma(n)}) \text{ with } j < k$$

and

$$(\xi_i \wedge \xi_j \wedge \phi_{\tau(1)} \wedge \cdots \wedge \phi_{\tau(q)} \wedge \xi_k \wedge \phi_{\tau(q+1)} \wedge \cdots \wedge \phi_{\tau(n)}) \text{ with } i < j.$$

Recall that when $i = 2$ in the first term and when $j = 3, k = 2$ in the second term, a coefficient of -1 must be introduced.

So we have

$$D \circ \bar{D}(\xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \phi_1 \wedge \cdots \wedge \phi_n) =$$

$$D\{\sum_\sigma \delta(\xi_i)(\phi_{\sigma(1)} \wedge \cdots \wedge \phi_{\sigma(p)} \wedge \xi_j \wedge \xi_k \wedge \phi_{\sigma(p+1)} \wedge \cdots \wedge \phi_{\sigma(n)}$$

$$+ \sum_\tau C(\xi_i, \xi_j)(\phi_{\tau(1)} \wedge \cdots \wedge \phi_{\tau(q)}) \wedge \xi_k \wedge \phi_{\tau(q+1)} \wedge \cdots \wedge \phi_{\tau(n)}\}$$

$$= D\{\sum_\sigma \xi_j \wedge \xi_k \wedge \delta(\xi_i)(\phi_{\sigma(1)} \wedge \cdots \wedge \phi_{\sigma(p)}) \wedge \wedge \phi_{\sigma(p+1)} \wedge \cdots \wedge \phi_{\sigma(n)}$$

$$+ \sum_\tau C(\xi_i, \xi_j)(\phi_{\tau(1)} \wedge \cdots \wedge \phi_{\tau(q)}) \wedge \xi_k \wedge \phi_{\tau(q+1)} \wedge \cdots \wedge \phi_{\tau(n)}\}$$

$$= \sum_\sigma C(\xi_i, \xi_j)(\delta(\xi_k)(\phi_{\sigma(1)} \wedge \cdots \wedge \phi_{\sigma(p)}) \wedge \phi_{\sigma(p+1)} \wedge \cdots \wedge \phi_{\sigma(n)})$$

$$+ \sum_\tau C(C(\xi_i, \xi_j)(\phi_{\tau(1)} \wedge \cdots \wedge \phi_{\tau(q)}), \xi_k(\phi_{\tau(q+1)} \wedge \cdots \wedge \phi_{\tau(n)})$$

which, after expanding the i, j, k terms of the unshuffles along with the signs mentioned above, is seen to equal the Jacobi relation, and hence is equal to 0. □

6. The classical strict Lie case

We examine the classical case in which Ξ is a Lie algebra and Φ is a Lie module over Ξ. Let us denote the action of Ξ on Φ by $\xi \cdot \phi$. We assume that we have a linear map $\partial : \Xi \to \Phi$ that interacts with the Lie module structure as follows:

$$\partial[\xi, \eta]_\Xi = \xi \cdot (\partial\eta) + \eta \cdot (\partial\xi)$$

where we have denoted the Lie bracket on Ξ by $[\cdot, \cdot]_\Xi$. As usual the Lie bracket on $\mathbb{L} = \Xi \oplus \Phi$ is given by

$$[x, y]_{\mathbb{L}} = \begin{cases} [x, y]_\Xi & \text{for } x, y \in \Xi \\ x \cdot y & \text{for } x \in \Xi, y \in \Phi \\ 0 & \text{for } x, y \in \Phi. \end{cases} \tag{1}$$

Similarly denote the Lie bracket on $Hom(\Lambda^*\Phi, \Phi)$ by $[\cdot, \cdot]_{Hom(\Phi)}$ (see Section 1) and on $Hom_k(\Lambda^*\Phi, \Xi)$ by $[\cdot, \cdot]_{Hom(\Xi)}$ (see Section 2).

Notice that this case is typical of the gauge structure which arises in fundamental physical theories such as Yang-Mills theory and basic gravitational theories. For the Yang-Mills case, the parameter space Ξ is the set of all smooth functions from the space-time M into the Lie algebra $\mathfrak{g}$ of the structure group G of the theory (for convenience of exposition, we assume that the principal bundle of the theory is trivial). The Lie bracket on the parameter space is the point-wise bracket of two such parameters. The fields of Yang-Mills theory are $\mathfrak{g}$-valued one-forms on M. Note that Berends, Burgers and van Dam denote the gauge transformation action of Ξ on Φ by $\{A, \Lambda\}$ (for $A \in \Phi, \Lambda \in \Xi$) rather than the notation $\Lambda \cdot A$ used above. In this case, this action is simply the covariant derivative of Λ relative to the connection A.

Similarly, when the Einstein-Hilbert action is utilized, the parameter space is the Lie algebra of all vector fields ξ on the space-time manifold M. Again, in Berends, Burgers and van Dam,the background metric η (Minkowski) is presumed and general metrics are written in the form $\eta + h$ for an appropriate symmetric tensor h. Thus the fields of the theory are symmetric tensors h. The action of a parameter ξ on a field h is the Lie derivative of h relative to the vector field ξ. The function δ is given by

$$(\delta h)_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + [(\partial_\rho h_{\mu\nu})\xi^\rho - h_{\rho\mu}(\partial^\rho \xi_\nu) - h_{\rho\nu}(\partial^\rho \xi_\mu)].$$

Details of these two standard examples may be found in Burgers' dissertation [Bur85].

Notice that using a bracket notation, $[\xi, \phi]_{\mathbb{L}} := \xi \cdot \phi$, for the action similar to that in Berends, Burgers and van Dam, the requirement that the bracket be a chain map with respect to ∂ is simply $\partial[\xi, \eta]_{\mathbb{L}} = [\xi, \partial\eta]_{\mathbb{L}} + [\eta, \partial\xi]_{\mathbb{L}}$. (We already require that $[\cdot, \cdot]_{\mathbb{L}}$ restricts to $[\cdot, \cdot]_{\Xi}$.)

Let us define the "gauge transformation" $\delta : \Xi \to Hom(\Lambda^*\Phi, \Phi)$ by

$$\delta(\xi)(\phi) = \begin{cases} \partial\xi & \text{for } \phi = 1 \\ \xi \cdot \phi & \text{for } \phi \in \Lambda^1\Phi = \Phi \\ 0 & \text{for } \phi = \phi_1 \wedge \cdots \wedge \phi_n \in \Lambda^n\Phi,\ n > 1. \end{cases} \tag{2}$$

Extend δ to $\hat{\delta} : Hom_k(\Lambda^*\Phi, \Xi) \to Hom(\Lambda^*\Phi, \Phi)$ by

$$\hat{\delta}(\pi)(\phi) = \begin{cases} \delta(\pi(\phi_1))(\phi_2) = \partial\pi(\phi) & \text{for } \phi_2 = 1 \\ \pi(\phi_1) \cdot \phi_2 & \text{for } \phi_2 \in \Lambda^1\Phi = \Phi \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Here $1 \in k \subset \mathcal{A}$ while ϕ denotes an arbitrary element of $\Lambda^*\Phi$ and $\Delta(\phi) = \sum \phi_1 \otimes \phi_2$.

The canonical bracket on $Hom_k(\Lambda^*\Phi, \Xi)$ that is induced by $\hat{\delta}$ and defined below will not satisfy the Jacobi identity in general. This bracket is given by

$$[\pi_1,\pi_2]_{Hom(\Xi)}(\phi)=\pi_1\circ\overline{\hat{\delta}(\pi_2)}(\phi)-\ pi_2\circ\overline{\hat{\delta}(\pi_1)}(\phi).$$

Here, $\overline{\hat{\delta}(\pi)}(\phi)=\hat{\delta}(\pi)(\phi_1)\wedge\phi_2=\delta(\pi(\phi_{11}))(\phi_{12})\wedge\phi_2$, and

$$\delta(\pi(\phi_{11}))(\phi_{12})\wedge\phi_2=\begin{cases}\partial(\pi(\phi_1)\wedge\phi_2 & \text{if } \phi_{12}=1\\ \pi(\phi_{11})\cdot\phi_{12})\wedge\phi_2 & \text{if } \phi_{12}\in\Lambda^1\Phi\\ 0 & \text{otherwise.}\end{cases}\tag{4}$$

In particular, if $\pi(\phi)=\xi(\phi)$ is defined to be the map with value ξ when $\phi=1\in k$ and 0 otherwise, then for $\phi=\sum(\phi_1\wedge\phi_2)$,

$$\overline{\hat{\delta}(\xi)}(\phi)=\begin{cases}\partial\xi\wedge\phi_2=\partial\xi\wedge\phi & \text{if } \phi_1=1\\ (\xi\cdot\phi_1)\wedge\phi_2 & \text{if } \phi_1\in\Lambda^1\Phi\\ 0 & \text{otherwise}\end{cases}\tag{5}$$

and so in $Hom_k(\Lambda^*\Phi,\Xi)$, the bracket

$$[\xi,\eta]_{Hom(\Xi)}(\phi)=(\xi\circ\overline{\hat{\delta}(\eta)})(\phi)-(\eta\circ\overline{\hat{\delta}(\xi)})(\phi)=0$$

because the coderivations in the definition of the bracket have image in $\Lambda^n\Phi$ with $n>0$.

It is important to note that the bracket on $Hom_k(\Lambda^*\Phi,\Xi)$ does not restrict to the original bracket on Ξ except in the abelian case; we must introduce the "correction" term C.

We continue with our construction and introduce the map

$$C:\Xi\otimes\Xi\to Hom_k(\Lambda^*\Phi,\Xi)$$

by defining $C(\xi,\eta)(\phi)=[\xi,\eta]_\Xi$ if $\phi=1$ and 0 otherwise. Here, $[\cdot,\cdot]_\Xi$ is the original Lie bracket on Ξ. Next, we must check that $\hat{\delta}C(\xi,\eta)=[\hat{\delta}(\xi),\hat{\delta}(\eta)]_{Hom(\Phi)}$ (notation as follows equation (1)).

So for $\phi\in\Lambda^*\Phi$, we have

$$\hat{\delta}C(\xi,\eta)(\phi)=\begin{cases}\partial[\xi,\eta]_\Xi & \text{if } \phi=1\\ [\xi,\eta]_\Xi\cdot\phi & \text{if } \phi\in\Lambda^1\Phi\\ 0 & \text{otherwise.}\end{cases}\tag{6}$$

On the other hand, we have

$$[\hat{\delta}(\xi),\hat{\delta}(\eta)]_{Hom(\Phi)}(\phi)=(\hat{\delta}(\xi)\circ\overline{\hat{\delta}(\eta)}-\hat{\delta}(\eta)\circ\overline{\hat{\delta}(\xi)})(\phi)$$

$$=\begin{cases}\hat{\delta}(\xi)(\partial\eta\wedge\phi_2)-\hat{\delta}(\eta)(\partial\xi\wedge\phi_2) & \text{if } \phi_1=1\\ \hat{\delta}(\xi)((\eta\cdot\phi_1)\wedge\phi_2)-\hat{\delta}(\eta)((\xi\cdot\phi_1)\wedge\phi_2) & \text{if } \phi_1\in\Lambda^1\Phi\\ 0 & \text{otherwise.}\end{cases}\tag{7}$$

The first term is non-zero only if $\phi_2 = 1$ in which case $\phi = 1$ and we have $\xi \cdot (\partial\eta) - \eta \cdot (\partial\xi)$ which is equal to $\partial[\xi,\eta]_\Xi$ by our original assumption on ∂. The second term is non-zero only for $\phi_2 = 1$ and $\phi_1 \in \Lambda^1\Phi$ and is then equal to $\xi \cdot (\eta \cdot \phi) - \eta \cdot (\xi \cdot \phi)$ which in turn is equal to $[\xi,\eta] \cdot \phi$ by the Lie module action of Ξ on Φ. Thus the BBvD hypothesis is satisfied.

Now we apply our Theorem 2 above to impose an sh-Lie structure on the graded vector space $\mathbb{L} = \{\mathbb{L}_n\}$ with $\mathbb{L}_0 = \Xi, \mathbb{L}_1 = \Phi$ and $\mathbb{L}_n = 0$ otherwise.

It is easy to see that our construction gives back the usual Lie algebra structure on the graded vector space $\mathbb{L}$, the semi-direct product of the Lie algebra Ξ and the module Φ.

7. On shell gauge symmetries

Up to this point we have focused primarily on unravelling the algebraic structure implicit in the BBvD hypothesis. This hypothesis is *trivially* satisfied for classical physical theories such as general relativity and Yang-Mills theories in the sense that the gauge symmetries of these physical theories satisfy the strict Lie version discussed in section 6. On the other hand, the BBvD hypothesis appears to be precisely the condition satisfied by the symmetries of "free differential algebras" which are useful in a careful description of the Sohnius-West model of supergravity, see for example [CDF91] and [CP95]. (Physicists refer to "free differential algebras" meaning differential graded commutative algebras which are free as graded commutative algebras.) Note that the latter paper shows that "free differential algebras" satisfy the BBvD hypothesis (see equation 4.16 in [CP95]) without any extra terms that vanish on shell. Consequently, some analysis such as the one developed in section 5 is required for a full understanding of the algebraic structure of these transformations.

Field dependent gauge symmetries appear in other field theories as well, including the class due to Ikeda [Ike94] and Schaller and Strobl [SS94] and employed by Cattaneo and Felder [CF99] to implement Kontsevich's deformation quantization [Kon97] referred to above. These field symmetries do not satisfy the BBvD hypothesis as we have described it above, but rather satisfy the BBvD hypothesis "on shell". In this section we outline how our work may be generalized so that in the next section we can show how to apply it to such field theories, illustrating this in terms of one due to Ikeda (and also that of Cattaneo and Felder).

First we explain what is meant when one says that a condition holds "on shell". In essence one means that the condition holds not for all the fields of the physical theory, but rather that it holds only for those fields which satisfy the field equations. In all the theories of interest here, the field equations are Euler-Lagrange equations. Such equations are obtained from the Lagrangian of the physical theory. In our case we assume that the Lagrangian is a polynomial in the components of both the fields and their derivatives. These components may be regarded as smooth functions on the space-time manifold M and consequently the Lagrangian is a mapping from the space Φ_0 of physical fields into $C^\infty M$ such that

$$L(\phi) = \mathcal{P}_L(\phi^a, \partial_I \phi^a)$$

where $\mathcal{P}_L(u^a, u^a_I)$ is a polynomial over $C^\infty M$ in the indeterminants u^a, u^a_I and where ϕ^a are the components of a typical field in Φ_0 (I is a symmetric multi-index). In that which follows, we identify u^a with u^a_I where I is empty. Similarly $\phi^a = \partial_I \phi^a$ where I is empty. The "action" of the physical theory is then the integral of the Lagrangian over the space-time manifold M. All of the theories discussed in Berends, Burgers and van Dam,the supergravity example mentioned above, and the example due to Ikeda, discussed more fully below, are polynomial Lagrangian field theories in the sense we have described above. The Euler operator E_a applied to the Lagrangian L produces the Euler-Lagrange differential operator $E_a L$ which acts on fields via

$$E_a L(\phi) = (-1)^{|I|} \quad \partial_I (\frac{\partial \mathcal{P}_L}{\partial u^a_I}(\partial_I \phi^b)).$$

Since the Lagrangian is polynomial in the components $\{\phi^a\}$ of the fields and their derivatives $\{\partial_I \phi^a\}$, the Euler-Lagrange differential operator is also a mapping from Φ_0 into $C^\infty M$ which factors through an appropriate polynomial over $C^\infty M$.

Observe that each homogeneous polynomial $\mathcal{P}(u^a_I)$ of degree k uniquely defines a symmetric multi-linear mapping β from $\mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_k$ into polynomials in $\bigcup_{i=1}^k \mathcal{U}_i$ such that

$$\mathcal{P}(u^a_I) = \beta(u^a_I, u^a_I, \cdots, u^a_I)$$

for appropriate indeterminates $\mathcal{U}_i = \{u(i)^{a_i}_I\}$. The polynomial $\mathcal{P}_L$ is a sum of homogeneous terms, each of which can be recovered from an appropriate symmetric multi-linear mapping by evaluating the multilinear mapping on the diagonal.

Consequently, each Lagrangian L uniquely identifies an element $\beta^L = \sum_i \beta_i^L$, where $\beta_i^L \in Hom(\wedge^i_{C^\infty M} \partial\Phi_0, C^\infty M)$, such that

$$L(\phi) = \sum_i \beta_i^L(\partial_I \phi^a, \partial_I \phi^a, \cdots, \partial_I \phi^a),$$

where $\partial\Phi_0$ denotes the vector space of the components of the fields and their derivatives. We refer to this identification as *polarization* and will be more precise in our algebraic formulation below. Similarly the Euler-Lagrange differential operator admits an analogous polarization.

It is probably useful to establish a dictionary relating our algebraic approach to field theory to more usual approaches. The algebra $\mathcal{A}$ is identified with the algebra $C^\infty M$ of smooth functions on the space-time M and Φ_0 with the space of all physical fields of the theory. This space of fields in simple cases is the space of all maps from M into a finite-dimensional vector space W. The module Φ is an algebraic way of formulating "jets" of fields and ∂ is the map which assigns the jet $\partial\phi = (\partial_I \phi^a) e_a^I$ to the field $\phi = \phi^a e_a$. Elements of $Hom(\wedge^*_{\mathcal{A}} \Phi, \mathcal{A})$ are identified with "polynomials" in the fields.

In our algebraic formulation, we let $\mathcal{A}$ denote any commutative associative algebra and let Φ_0 denote an arbitrary $\mathcal{A}$-module freely and finitely generated over $\mathcal{A}$ with basis $\{e_a\}$. Working locally, we assume the existence of a finite number of derivations ∂_μ of $\mathcal{A}$ which admit extensions as $\mathcal{A}$-derivations of Φ_0 in the sense that for each μ, $\partial_\mu(e_a) = 0$ and $\partial_\mu(f\phi) = f\partial_\mu\phi + (\partial_\mu f)\phi$, for $f \in \mathcal{A}, \phi \in \Phi_0$. For each symmetric multi-index $I = (i_1, i_2, \cdots, i_k)$, let $\partial_I = \partial_{i_1} \circ \cdots \circ \partial_{i_k}$ and let Φ denote the $\mathcal{A}$-module freely generated by symbols $\{e_I^a\}$ so that

$$\Phi = \{\phi_I^a e_a^I | \phi_a^I \in \mathcal{A}\}.$$

In this context, L and the Euler-Lagrange differential operators $E_a L$ are identified with their polarizations which are special elements of $Hom(\wedge^*_{\mathcal{A}} \Phi, \mathcal{A})$ where $\wedge^*_{\mathcal{A}} \Phi$ is the free nilpotent cocommutative coalgebra generated by Φ over $\mathcal{A}$.

More precisely, when we say that $L : \Phi_0 \longrightarrow \mathcal{A}$ is a polynomial Lagrangian, we mean that there is a unique $\beta^L \in Hom(\wedge_{\mathcal{A}} \Phi, \mathcal{A})$ such that

$$L(\phi) = \sum_i \beta_i^L(\partial\phi, \partial\phi, \cdots, \partial\phi),$$

where $\beta_i^L \in Hom(\wedge^i_{\mathcal{A}} \Phi, \mathcal{A})$ is homogeneous and ∂ is the mapping from Φ_0 into Φ defined by $\partial\phi = \partial_I \phi^a e_a^I$.

Here, of course, we mean that $\beta^L = \sum_i \beta_i^L$ where, for each i, β_i^L can only be nonzero on $\wedge^i_{\mathcal{A}} \Phi$, i.e., for each i, β_i^L is multilinear and symmetric having the property that when it is evaluated on $\partial\phi \wedge \cdots \wedge \partial\phi$

one obtains precisely that term in $\mathcal{P}_L(\phi^a, \partial_I \phi^a)$ of degree i. To obtain all the terms in $L(\phi)$, one must sum over all the homogeneous terms which appear in the polynomial $\mathcal{P}_L$ which determines L. It is possible to recover the mapping $E_a L : \Phi_0 \longrightarrow \mathcal{A}$ from an element of $Hom(\wedge_{\mathcal{A}}\Phi, \mathcal{A})$ in a similar manner. Consequently, in that which follows, we identify L with β^L and we regard both L and $E_a L$ as elements of $Hom(\wedge_{\mathcal{A}}\Phi, \mathcal{A})$.

In this formulation, the "shell" is the subset Σ of Φ_0 defined by

$$\Sigma = \{\phi \in \Phi_0 | \quad E_a L(diag(\partial\phi)) = 0\},$$

where the diagonal mapping $diag : \Phi \longrightarrow \wedge^*\Phi$ is defined by

$$diag(\phi) = \sum_p \phi^p,$$

and where

$$\phi^p = (\phi \wedge \phi \wedge \cdots \wedge \phi) \in \wedge^p_{\mathcal{A}}\Phi.$$

It is required that $E_a L \in Hom(\wedge^*\Phi, \mathcal{A})$ be zero only on the diagonal as the restriction of $E_a L$ to the diagonal agrees with the polynomial counterpart of $E_a L$ and it is the zero set of this latter function which defines the solution space of the usual Euler-Lagrange operator. Note that Σ is *not* a subspace of $\wedge^*\Phi$.

Define a subspace $\mathcal{I}$ of $Hom(\wedge^*\Phi, \mathcal{A})$ by

$$\mathcal{I} = \{f \in Hom(\wedge^*\Phi, \mathcal{A}) | f(diag(\partial\phi)) = 0, \phi \in \Sigma\}.$$

Similarly, define a subspace $\mathcal{N}$ of $Hom(\wedge^*\Phi, \Phi)$ by

$$\mathcal{N} = \{\nu \in Hom(\wedge^*\Phi, \Phi) | \nu(diag(\partial\phi)) = 0, \phi \in \Sigma\}.$$

We say that $f \in Hom(\wedge^*\Phi, \mathcal{A})$ and $\nu \in Hom(\wedge^*\Phi, \Phi)$ vanish "on shell" iff f and ν are in $\mathcal{I}$ and $\mathcal{N}$, respectively.

Elements of $\mathcal{I}$ are "polynomials" such as $E_a L$ which vanish "on shell". The "polynomials" referred to here are actually mappings from Φ_0 to $\mathcal{A}$ which factor through polynomials over $\mathcal{A}$ in the indeterminates $\{u^a_I\}$ as in our description of the Lagrangian L above. $Hom(\wedge^*_{\mathcal{A}}\Phi, \Phi)$ plays the role of vector fields with coefficients from $Hom(\wedge^*_{\mathcal{A}}\Phi, \mathcal{A})$, and $\mathcal{N}$ plays the role of the space of vector fields whose coefficients vanish on Σ.

At this point, we generalize the BBvD hypothesis as follows. We say that the k-linear mapping $\delta : \Xi \longrightarrow Hom(\wedge^*_{\mathcal{A}}\Phi, \Phi)$ satisfies the *generalized Berends, Burgers and van Dam hypothesis*, denoted gBBvD, iff there exists a skew-symmetric k-bilinear mapping $C : \Xi \times \Xi \longrightarrow Hom(\wedge^*_{\mathcal{A}}\Phi, \Xi)$ and an extension $\hat{\delta}$ of δ to $Hom(\wedge^*_{\mathcal{A}}\Phi, \Xi)$ such that

$$[\delta(\xi), \delta(\eta)] - \hat{\delta}(C(\xi, \eta)) \in \mathcal{N}.$$

for all $\xi, \eta \in \Xi$. Thus the BBvD hypothesis of Section 3 holds "on shell". A consequence of this hypothesis is that there exists a skew-symmetric mapping $\nu : \Xi \times \Xi \longrightarrow \mathcal{N}$ such that

$$[\delta(\xi), \delta(\eta)] = \hat{\delta}(C(\xi, \eta)) + \nu(\xi, \eta).$$

Utilizing this mapping C, one can define a bracket on $\mathrm{Hom}_k(\Lambda^*\Phi, \Xi)$ analogous to that defined before in the presence of the BBvD hypothesis:

$$[\pi_1, \pi_2] := \pi_1 \odot \hat{\delta}(\pi_2) - \pi_2 \odot \hat{\delta}(\pi_1) + C(\pi_1, \pi_2).$$

Injectivity of $\hat{\delta}$ is not easily obtained and seems to be needed to obtain a proof of the Jacobi identity. Thus, in general the bracket on $\mathrm{Hom}_k(\Lambda^*\Phi, \Xi)$ will not satisfy the Jacobi identity.

On the other hand, we can use the calculations of the proof of Theorem 1 to show that

$$[\hat{\delta}(\pi_1), \hat{\delta}(\pi_2)] - \hat{\delta}([\pi_1, \pi_2]) \in \mathcal{N}$$

for all π_1, π_2 in $\mathrm{Hom}_k(\Lambda^*\Phi, \Xi)$. By using the calculations in the proof of Theorem 2, it is easy to show that:

$$\begin{aligned} D \circ \overline{D}(\xi_1 \wedge \xi_2 \wedge \phi_1 \wedge \cdots \wedge \phi_n) = \\ = -[\delta(\xi_1), \delta(\xi_2)](\phi_1 \wedge \cdots \wedge \phi_n) + \hat{\delta}(C(\xi_1, \xi_2))(\phi_1 \wedge \cdots \wedge \phi_n) \\ = \nu(\xi_1, \xi_2)(\phi_1 \wedge \cdots \wedge \phi_n) \end{aligned}$$

and that

$$D \circ \overline{D}(\xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \phi_1 \wedge \cdots \wedge \phi_n) = Jacobi(\xi_1, \xi_2, \xi_3)(\phi_1 \wedge \cdots \wedge \phi_n)$$

where

$$Jacobi(\xi_1, \xi_2, \xi_3) = ([[\xi_1, \xi_2], \xi_3] - [[\xi_1, \xi_3], \xi_2] + [[\xi_2, \xi_3], \xi_1]).$$

In this latter equation we have used the notation $[\xi, \eta]$ in place of $C(\xi, \eta)$.

It follows from these equations that $D \circ \overline{D}$ is zero "on shell" provided that both the generalized BBvD hypothesis holds and that $Jacobi(\xi, \eta, \zeta)$ is zero on shell for arbitrary constants $\xi, \eta, \zeta \in Hom(\wedge^*_{\mathcal{A}}\Phi, \Xi)$.

In the example due to Ikeda [Ike94] discussed in detail in Section 8, it is easy to prove that $Jacobi(\xi_1, \xi_2, \xi_3) = 0$ (not just zero "on shell") using the equation immediately prior to equation 2.10 in his paper. Consequently the gauge symmetries of this Poisson σ-model satisfy the postulates of an sh-Lie algebra "on shell".

8. A Σ-model example

In Ikeda's paper [Ike94], there is a finite dimensional vector space V with basis $\{T_A\}$ which later we will show is the dual of a Poisson manifold. We do this via a generalization of the classical Kirillov-Kostant bracket which exhibits the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ as a Poisson manifold.

In our analysis of Ikeda's example, our space Ξ is the space of maps $Maps(\Sigma, V)$ and the space Φ_0 is the set of ordered pairs $\phi = (\psi, h)$ where:

(1) ψ is a mapping from a given two-dimensional manifold Σ into the dual V^* of the vector space V, and

(2) h is a mapping from the same manifold Σ to $T^*\Sigma \otimes V$, which in fact is required to be a section of the vector bundle $T^*\Sigma \otimes V \longrightarrow \Sigma$. These mappings are denoted locally by $\psi(x) = \psi_A(x)T^A$ and $h(x) = h^A_\mu(dx^\mu \otimes T_A)$, where $\{T_A\}$ is a basis of V and $\{T^A\}$ is the basis of V^* dual to $\{T_A\}$.

For the most part, our exposition follows that of Ikeda, although we use the notation $\phi = (\psi, h)$ for the fields of the theory whereas Ikeda's notation for the fields is (ϕ, h). We also denote Ikeda's vector space M by V. As is the case earlier in the paper, the space Φ denotes the $\mathcal{A} = C^\infty M$ module whose elements are $\phi^a_I e^I_a$ where $\{e^I_a\}$ is a basis of the module and $\phi^a_I \in \mathcal{A}$. This formulation is our algebraic description of the jet bundle of the vector bundle whose sections are the fields Φ_0. Ikeda would denote ϕ^a_I as $\partial_I \phi^a$. Observe that Φ_0 may be identified as a subspace of Φ.

There is a parallel development to Ikeda's work in Cattaneo and Felder [CF99] in which Σ is a 2-dimensional disc and the target (denoted by M in Cattaneo and Felder) is an arbitrary Poisson manifold. It is not hard to see that the ordered pairs (ψ, h) of Ikeda may in fact be interpreted in a manner similar to that in the exposition of Cattaneo and Felder where $\psi : \Sigma \longrightarrow M$ is an arbitrary smooth mapping (ψ is denoted by X in Cattaneo and Felder) and h is a section (denoted by η in Cattaneo and Felder) of the bundle $\psi^*(T^*M) \otimes T^*\Sigma \longrightarrow \Sigma$ (notice that the factors in their tensor product are reversed from the conventions used in our description of Ikedas' results). In their exposition the section h may be written as $h(x) = h_{i,\mu}(x)(dx^i \otimes du^\mu)$ where $\{dx^i\}$ is a basis of $T^*_{\psi(x)}M$, which, in the case M is a vector space V, may be identified with a fixed basis $\{T^A\}$ of $T^*_0 V = V^*$.

When one compares these two approaches, one sees that Ikeda's target space is the vector space we have called V^* while Σ is an arbitrary 2-dimensional manifold, whereas for Cattaneo and Felder Σ is a disc

D and the target space M is a general Poisson manifold. The parallel between the two is closer than one might initially expect since Ikeda uses the vector space V to generate a Poisson structure on V^*.

Ikeda proceeds to investigate possible gauge symmetries $\delta(c)$ *before* looking for Lagrangians. The gauge symmetry mapping δ is defined locally, in this theory, as follows. Let $\mathcal{P}$ denote the commutative polynomial algebra generated by the basis $\{T_A\}$. Let π_A, π^A_μ, π^A denote the projections defined by $\pi_A(\phi) = \pi_A(\psi, h) = \pi_A(\psi) = \psi_A$, $\pi^A_\mu(\phi) = \pi^A_\mu(\psi, h) = \pi^A_\mu(h) = h^A_\mu$ and $\pi^A(c) = c^A$, respectively. Consider arbitrary polynomials $\{W_{AB}\}$ in $\mathcal{P}$ and define the components of $\delta(c)(\phi)$ by

$$\pi^A_\mu(\delta(c)(\phi)) = \partial_\mu c^A + \frac{\partial W_{BD}(\psi)}{\partial T_A} h^B_\mu c^D$$

and

$$\pi_A(\delta(c)(\phi)) = W_{BA}(\psi)c^B.$$

Here $W_{AB}(\psi)$ is a concise notation for the polynomial W_{AB} evaluated by replacing the generators $\{T_A\}$ by the correponding components $\{\psi_A\}$ of ψ, that is, $W_{AB} = W^a_{AB}T_a$ and $W_{AB}(\psi) = W^a_{AB}\psi_a$ where a is a symmetric multi-index, $\psi_a = \psi_{A_1}\psi_{A_2}\cdots\psi_{A_n}$ and similarly for T_a.

Notice that, in case V is a Lie algebra and $W_{AB}(T) = [T_A, T_B] = f^C_{AB}T_C$, the polynomials W_{AB} define the Lie algebra structure on the vector space V with structure constants $\{f^C_{AB}\}$. This then induces a Lie algebra structure on the parameter space Ξ of all mappings c from Σ into V, as one expects in traditional Yang-Mills theory. In this case, the ψ-component of $\delta(c)$ is the coadjoint action of the parameter space Ξ on the space of maps from Σ into V, while the h-component is simply the "covariant derivative" of c relative to the connection defined by the gauge field h. Thus, by introducing more general polynomials W_{AB}, Ikeda is introducing a generalization of ordinary gauge theory by requiring that the gauge symmetries be defined via the polynomials W_{AB}. For this generalization to work, Ikeda imposes restrictions on the polynomials W_{AB} which amount to making $\mathcal{P}$ a Lie algebra, hence his terminology of 'non-linear Lie algebra'. In order to obtain an algebraic structure on $\mathcal{P}$ analogous to the usual Lie structure required in gauge theory, Ikeda's bracket is defined on generators of $\mathcal{P}$ by

$$[T_A, T_B] = W_{AB} \in \mathcal{P}$$

and extended to all of $\mathcal{P}$ via the Leibniz rule: $[T_A,\ ]$ and $[\ ,T_B]$ are derivations of the commutative algebra $\mathcal{P}$. Ikeda requires that these polynomials satisfy conditions which make $\mathcal{P}$ a Poisson algebra. Thus the polynomials $\{W_{AB}\}$ in $\mathcal{P}$ are subject to skew-symmetry: $W_{AB} =$

$-W_{BA}$ and an appropriate generalization of the usual coordinate form of the Jacobi condition:

$$W_{AD}\frac{\partial W_{BC}}{\partial T_D}+W_{BD}\frac{\partial W_{CA}}{\partial T_D}+W_{CD}\frac{\partial W_{AB}}{\partial T_D}=0.$$

To see V^* as a Poisson manifold, we will imbed V in V^{**} as the linear functionals and thus regard the algebra $\mathcal{P}$ as the subalgebra of $C^\infty(V^*)$ generated by the basis $\{T_A\}$.

Regarding T_A's as functions on V^*, we have a bi-vector field

$$W_{AB}\frac{\partial}{\partial T_A}\wedge\frac{\partial}{\partial T_B}$$

on V^*.

This makes V^* a Poisson manifold with

$$\{f,g\}:=W_{AB}\frac{\partial f}{\partial T_A}\wedge\frac{\partial g}{\partial T_B}$$

for $f,g\in C^\infty(V^*)$. Now notice that, using Ikeda's notation as defined above, we have for each c that $\delta(c)$ is a mapping from Φ_0 to Φ_0. Since Φ_0 is a vector space, it follows that with any reasonable topology on Φ_0 one can identify the tangent space of Φ_0 at a point $\phi\in\Phi_0$ with Φ_0 itself. Thus maps from Φ_0 into Φ_0 may be regarded as vector fields on Φ_0. Recall that $\delta(c)$ is a vector field on the space Φ_0 of fields. Thus $\delta(c)(\phi)$ is a tangent vector to Φ_0 at ϕ.

By an obvious abuse of notation one may write:

$$\delta(c)(\phi)=(W_{BA}(\psi)c^B)\frac{\partial}{\partial\psi_A}+(\partial_\mu c^A+\frac{\partial W_{BD}(\psi)}{\partial T_A}h^B_\mu c^D)\frac{\partial}{\partial h^A_\mu}. \tag{8}$$

Now the components of $\delta(c)(\phi)$ as defined in equation (8) are polynomials in the components of ϕ. Consequently, in conformity with our conventions in section 7, we can identify $\delta(c)$ with the unique element of $Hom(\wedge^*\Phi,\Phi)$ whose value at $diag(\partial\phi)$, for $\phi\in\Phi_0$, gives $\delta(c)(\phi)$ as defined by Ikeda.

The usual Lie bracket of the vector fields $\delta(c_1)$ and $\delta(c_2)$ as defined by equation 8 corresponds to our Lie structure on $Hom(\wedge^*\Phi,\Phi)$. Using his brackets, Ikeda finds that the ψ component of $[\delta(c_1),\delta(c_2)](\phi)$ is given by

$$[\delta(c_1),\delta(c_2)](\psi)=\delta(c_3(\psi))(\psi)$$

where

$$\pi_A(c_3(\psi))=\frac{\partial W_{BD}}{\partial T_A}(\psi)c_1^Bc_2^D. \tag{9}$$

We see that the Lie bracket of $[\delta(c_1), \delta(c_2)]$ is not of the form $\delta(c)$ where c_3 is a gauge parameter independent of the fields ϕ but rather the gauge parameter c depends on c_1, c_2 and on the field ψ. Thus one does not have closure on the original space of gauge parameters $Maps(\Sigma, V) = \Xi$. We are forced to enlarge the space of gauge parameters Ξ to include mappings from Φ to Ξ.

In Ikeda's context, these mappings are polynomials in the components of the fields and their derivatives. Consequently, they are identified with elements of $\mathrm{Hom}_k(\Lambda^*\Phi, \Xi)$ in our formulation.

If we had only the fields ψ to deal with, the BBvD hypothesis would be satisfied and we would be able to apply the ideas in earlier sections to describe Ikeda's algebra of gauge transformations as an sh-Lie algebra on a graded space with Ξ in degree zero. However, the h-component transforms more subtly. To handle this, Ikeda makes the definition

$$D_\mu \psi_A = \partial_\mu \psi_A + W_{AB}(\psi) h^B_\mu. \tag{10}$$

(The resemblance to a covariant derivative is formal; it is not yet understood as arising in an obvious manner from a "representation" of the nonlinear Lie algebra defined by Ikeda.) He then calculates

$$[\delta(c_1), \delta(c_2)](h) = \delta(c_3(\psi))(h) - \left(\frac{\partial^2 W_{CD}}{\partial\psi_A \partial\psi_B}(D_\mu \psi_B) c_1^C c_2^D\right)\frac{\partial}{\partial h^A_\mu}. \tag{11}$$

Thus the BBvD hypothesis fails, but the generalized hypothesis holds. When closure on the original space of V-valued gauge parameters is lost, physicists speak of an 'open algebra'.

Having established his gauge algebra and potential gauge symmetries, Ikeda then searches for an appropriate Lagrangian. Up to a total divergence, the Lagrangian of Ikeda's theory is

$$\mathcal{L} = \epsilon^{\mu\nu}\{h^A_\mu D_\nu \psi_A - \frac{1}{2} W_{AB}(\psi) h^A_\mu h^B_\nu\}.$$

This includes self-interacting terms for the generalized gauge fields h along with a minimal coupling of the scalar field ψ through the generalized covariant derivative defined in equation (10) above. The tensor $\epsilon^{\mu\nu}$ is the area element which is assumed to be present on Σ.

Ikeda really works with an equivalent Lagrangian which differs from the one given above by a divergence, although the physical content of the Lagrangian defined above is clearer.

Ikeda shows that for his equivalent Lagrangian

$$\mathcal{L}(\phi, \partial\phi) = \mathcal{L}(\psi, h, \partial\psi, \partial h),$$

the function $\delta(c)(\mathcal{L})$ is a divergence for all parameters c. This is precisely the property physicists require in order to call δ a gauge symmetry.

The field equations of the Lagrangian are

$$D_\mu \psi_A = 0 \qquad R^A_{\mu\nu} = 0$$

where $R^A_{\mu\nu}$ is the "generalized" curvature

$$R^A_{\mu\nu} = \partial_\mu h^A_\nu - \partial_\nu h^A_\mu + \frac{\partial W_{BC}}{\partial T_A}(\psi) h^B_\mu h^C_\nu$$

of the "generalized gauge field" $h = h^A_\mu (dx^\mu \otimes T_A)$.

The coefficient of the last term on the right-hand side of equation (11) is polynomial in the components of $\phi \in \Phi_0$ and their derivatives and (as in section 7) determines a unique bilinear mapping ν from $\Xi \times \Xi$ into $\mathcal{N} = \{\nu \in Hom(\wedge^* \Phi, \Phi) | \nu(diag(\partial\phi)) = 0, \phi \in \Sigma\}$ such that

$$[\delta(c_1), \delta(c_2)] = \delta(C(c_1, c_2)) + \nu(c_1, c_2) \tag{12}$$

where $C(c_1, c_2) = c_3 : \wedge^* \Phi \longrightarrow \Xi$ is defined by equation (9). This latter property (12) is the one we have referred to above as the gBBvD hypothesis.

A similar analysis applies to the Lagrangian of Cattaneo and Felder. Thus Ikeda and Cattaneo and Felder provide examples of field theories which satisfy the generalized BBvD hypothesis and it is this condition which we have assumed in sections 7 and 8. The gauge symmetries of these theories require a modification of the sh-Lie structure one obtains from the gauge structures of field theories satisfying the BBvD hypothesis.

References

[BBvD84] F.A. Berends, G.J.H. Burgers, and H. van Dam, *On spin three selfinteractions*, Z. Phys. C **24** (1984), 247–254.

[BBvD85] F.A. Berends, G.J.H. Burgers, and H. van Dam, *On the theoretical problems in constructing intereactions involving higher spin massless particles*, Nucl.Phys.B **260** (1985), 295–322.

[BBvD86] F.A. Berends, G.J.H. Burgers, and H. van Dam, *Explicit construction of conserved currents for massless fields of arbitrary spin*, Nucl.Phys.B **271** (1986), 429–441.

[Bur85] G.J.H. Burgers, *On the construction of field theories for higher spin massless particles*, Ph.D. thesis, Rijksuniversiteit te Leiden, 1985.

[CDF91] L. Castellani, R. D'Auria, and P. Fré Supergravity and superstrings, Vol. 2, World Scientific , 1991.

[CF99] A. Cattaneo and G. Felder, *A path integral approach to the Kontsevich quantization formula*, `math.QA/9902090` .

[CP95] L. Castellani and A. Perotto , *Free differential algebras: their use in field theory and dual formulation*, Lett. Math. Phys. 38 (1996) 321-330, hep-th/9509031.

[Ger62] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. Math. **78** (1962), 267–288.

[Ike94] N. Ikeda, *Two-dimensional gravity and nonlinear gauge theory*, Ann. Phys. **235** (1994), 435–464.

[Ike01] N. Ikeda, *A deformation of three dimensional BF theory*, hep-th/0010096.

[Kon97] M. Kontsevich, *Deformation quantization of Poisson manifolds, I*, preprint, IHES, 1997, hep-th/9709040.

[LM95] T. Lada and M. Markl, *Strongly homotopy Lie algebras*, Comm. in Algebra (1995), 2147–2161.

[LS93] T. Lada and J.D. Stasheff, *Introduction to sh Lie algebras for physicists*, Intern'l J. Theor. Phys. **32** (1993), 1087–1103.

[SS94] P. Schaller and T. Strobl, *Poisson structure induced (topological) field theories*, Modern Phys. Lett. A **9** (1994), 3129–3136.

[Sta93] J. D. Stasheff, *The intrinsic bracket on the deformation complex of an associative algebra*, JPAA **89** (1993), 231–235, Festschrift in Honor of Alex Heller.

Department of Mathematics, North Carolina State University, Raleigh NC 27695

E-mail address: fulp@math.ncsu.edu

Department of Mathematics, North Carolina State University, Raleigh NC 27695

E-mail address: lada@math.ncsu.edu

Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250, USA

E-mail address: jds@math.unc.edu