A SMALL OPEN-CLOSED HOMOTOPY ALGEBRA (OCHA)

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Abstract. We consider a particular finite dimensional example of an L_∞ algebra in which a 2-dimensional Lie algebra acts on a 1-dimensional vector space in a non-trivial non-Lie manner. In order to understand the nature of this action, we show that this algebra is in fact an example of an open-closed homotopy algebra.

1. Introduction

In [1], a non-trivial L_∞ algebra structure on a finite dimensional 2-graded vector space which was discovered by M. Daily was discussed in detail. The structure of that algebra entailed a 2-dimensional Lie algebra V_0 acting on a 1-dimensional vector space V_1. The nature of this action is the topic of this article. A possible structure for this action is that of V_1 being an L_∞ module over the Lie algebra V_0. Such an action requires a collection of operations $\eta_k : V_0^{\otimes(k-1)} \otimes V_1 \to V_1$ subject to compatibility relations; see [4] for details.

Two other candidates for understanding the action of V_0 on V_1 are that of an A_∞ algebra over an L_∞ algebra and that of an open-closed homotopy algebra as developed by Kajiura and Stasheff [3]. These structures are given by operations $\eta_{p,q} : V_0^{\otimes p} \otimes V_1^{\otimes q} \to V_1$ subject to compatibility relations, where $p \geq 0, q \geq 1$ for an A_∞ algebra over an L_∞ algebra, and $p \geq 0, q \geq 0$ for an open-closed homotopy algebra. These actions may also be described by a coderivation D on the coalgebra $S^c(\downarrow V_0) \otimes T^c(\downarrow V_1)$ with $D^2 = 0$ [3],[2]. Here, $S^c(V_0)$ is the cocommutative coalgebra on V_0 , $T^c(V_1)$ is the tensor coalgebra on V_1, and $\downarrow$ is the desuspension isomorphism of graded vector spaces.

Our main result will show that the L_∞ algebra mentioned above is in fact an example of an open-closed homotopy algebra. The other two types of action are not possible because of the presence of a non zero operation $\eta_{1,0}$.

We will review the definition of L_∞ algebras in Section 2 and provide explicit details of the example. In Section 3, we will recall the definition of an open-closed homotopy algebra and verify that the example satisfies the relations.

2. An L_∞ algebra

We begin by recalling the definition of an L_∞ algebra [4].

Definition 1. *An L_∞ algebra structure on a graded vector space V is a collection of skew symmetric linear maps $l_n : V^{\otimes n} \to V$ of degree $2-n$ that satisfy the relations*

$$\sum_{i+j=n+1} \sum_{\sigma} (-1)^{\sigma} (-1)^{e(\sigma)} (-1)^{i(j-1)} l_j(l_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}) = 0$$

where $(-1)^{\sigma}$ is the sign of the permutation, $e(\sigma)$ is the product of the degrees of the permuted elements, and σ is taken over all $(i, n-i)$ unshuffles.

This is the cochain complex point of view; for chain complexes, require the maps l_n to have degree $n-2$.

Now consider the graded vector space $V = V_0 \oplus V_1$ where V_0 has basis $< v_1, v_2 >$ and V_1 has basis $< w >$. Then V may be given an L_∞ algebra structure by defining [1]

$$l_1(v_1) = l_1(v_2) = w$$

$$l_2(v_1 \otimes v_2) = v_1, l_2(v_1 \otimes w) = w$$

$$l_n(v_2 \otimes w^{\otimes n-1}) = C_n w = (-1)^{\frac{(n-2)(n-3)}{2}}(n-3)!w, n \geq 3.$$

In other words, (V, l_1) is a cochain complex and the maps l_n have degree $2-n$, are extended to all of $V^{\otimes n}$ by graded skew symmetry, and are defined to be equal to 0 on any elements not mentioned above. Also note that (V_0, l_2) is a two dimensional Lie algebra.

There is an equivalent description of L_∞ algebras given by a degree 1 coderivation D on the on the coalgebra $S^c(\downarrow V)$ with $D^2 = 0$ [4], [5] . We will translate the L_∞ algebra data above into this context in order to be compatible with the OCHA data in the next section.

We may apply the desuspension operator, $\downarrow$, to the data above to obtain a collection of degree one graded symmetric linear maps $\hat{l_n} : W^{\otimes n} \to W$ given by $\hat{l}_n = (-1)^{\frac{n(n-1)}{2}} \downarrow \circ l_n \circ \uparrow^{\otimes n}$[4]. Here, $W = W_{-1} \oplus W_0$ with W_{-1} isomorphic to V_0 and W_0 isomorphic to V_1. Let x_i correspond to v_i and y correspond to w under these isomorphisms. We may then describe the $\hat{l}_n$'s explicitly by

$$\hat{l}_1(x_1) = \hat{l}_1(x_2) = y$$

$$\hat{l}_2(x_1 \otimes x_2) = x_1, \hat{l}_2(x_1 \otimes y) = y$$

$$\hat{l}_n(x_2 \otimes y^{n-1}) = C'_n y = (-1)^n (n-3)!y$$

The signs in the above equation result from the definition of $\hat{l}_n$ in terms of l_n, the definition of l_n in this particular example, and from applying the map $\uparrow^{\otimes n}$ to the element $x_2 \otimes y^{n-1}$ using the fact that the degree of x_2 is -1 and the degree of y is 0.

For example,

$$\hat{l}_n(x_2, y, \ldots, y) = (-1)^{\frac{n(n-1)}{2}} \downarrow \circ l_n \circ \uparrow^{\otimes n} (x_2, y, \ldots, y)$$

which, after applying the map $\uparrow^{\otimes n}$ and noting that the degree of y is 0 and the degree of x_2 is -1, and that $\uparrow x_2 = v_2$ and $\uparrow y = w$,

$$= (-1)^{\frac{n(n-1)}{2}}(-1)^{n-1} \downarrow \circ l_n(v_2, w, \ldots, w)$$

$$= (-1)^{\frac{n(n-1)}{2}}(-1)^{n-1}(-1)^{\frac{(n-2)(n-3)}{2}}(n-3)! \downarrow w = (-1)^n(n-3)!y.$$

The last equality results from the fact that $\frac{n(n-1)}{2} + (n-1) + \frac{(n-2)(n-3)}{2}$ has the same parity as n.

These maps $\hat{l}_n$ have degree $+1$ and may be extended to coderivations on $S^c(W)$ and added together to give the differential D on the cocommutative coalgebra $S^c(W)$.

3. An open-closed homotopy algebra

We next recall the definition of an open-closed homotopy algebra (OCHA) as introduced by Kajiura and Stasheff [3].

Definition 2. *An open-closed homotopy algebra* $(\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_o, l, \eta)$ *consists of an* L_∞ *algebra* $(\mathcal{H}_c, l)$ *and a family of degree* $+1$ *maps* $\{\eta_{p,q} : \mathcal{H}_c^{\otimes p} \otimes \mathcal{H}_o^{\otimes q} \to \mathcal{H}_o\}$ *for* $p, q \geq 0$ *such that*

$$0 = \sum_\sigma (-1)^{\epsilon(\sigma)} \eta_{1+r,m}(l_p(c_{\sigma(1)}, \ldots, c_{\sigma(p)}), c_{\sigma(p+1)}, \ldots, c_{\sigma(n)}; o_1, \ldots, o_m)$$

$$+ \sum_\sigma (-1)^{\mu_{p,i}(\sigma)} \eta_{p,i+1+j}(c_{\sigma(1)}, ..c_{\sigma(p)}; o_1, .., o_i, \eta_{r,s}(c_{\sigma(p+1)}, .., c_{\sigma(n)}; o_{i+1}, .., o_{i+s}), o_{i+s+1}, .., o_m)$$

where the second sum is taken over $i + s + j = m$, σ *ranges over all* $(p, n-r)$ *unshuffles, and* $n, m \geq 0$.

The sign exponent is given by

$$\mu_{p,i}(\sigma) = \epsilon(\sigma) + (c_{\sigma(1)} + .. + c_{\sigma(p)}) + (o_1 + .. + o_i) + (o_1 + .. + o_i)(c_{\sigma(p+1)} + .. + c_{\sigma(n)})$$

where we indicate the degree of an element by its name, and $\epsilon(\sigma)$ is the product of the degrees of the permuted elements.

We will show that the following example of a "small" L_∞ algebra has the structure of an OCHA.

Theorem 3. *The graded vector space* $W = W_{-1} \oplus W_0$ *together with the maps* $\{\hat{l}_n\}$ *has the structure of an open-closed homotopy algebra.*

Proof. We let $\mathcal{H}_c = W_{-1}$ and $\mathcal{H}_o = W_0$ and define $\eta_{p,q} = \frac{1}{q!}\hat{l}_{p+q}$. Recall that W_{-1} together with $\hat{l}_2$ restricted to $W_{-1} \otimes W_{-1}$ is a Lie algebra, so the requirement that $\mathcal{H}_c$ be an L_∞ algebra is satisfied. We next show that the maps $\eta_{p,q}$ satisfy the relations in the definition by evaluating those terms on all possible inputs from W. We first observe that by the definition of $\hat{l}_n$, all $\eta_{0,q} = 0$. From this, it is immediate that any element of $\mathcal{H}^{\otimes n}$ with only a single element from $\mathcal{H}_c$ will trivially satisfy the requisite relations.

The next case to consider is an element of the form $x_1 \otimes x_1 \otimes y^m$. Because $\hat{l}_2(x_1 \otimes x_1) = 0$, we need only consider the second sum in the relation. The only possible non-zero term occurs in the expression $\eta_{1,1}(\eta_{1,0}(x_1), x_1) = \hat{l}_2(\hat{l}_1(x_1), x_1)) = \hat{l}_2(y, x_1) = y$. However, this term occurs again with opposite sign because there are two (1,1) unshuffles of $x_1 \otimes x_1$ and the Koszul sign is -1 for the second one. Consequently, the relation is satisfied in this case.

A similar situation occurs with the case of elements of the form $x_2 \otimes x_2 \otimes y^m$. However, we have here non-trivial terms of the form (each $y_i = y$)

$$\begin{aligned} &\eta_{1,m-s+1}(x_2; y_1, .., y_i, \eta_{1,s}(x_2; y_{i+1}, .., y_{i+s}), y_{i+s+1}, ..y_m) \\ &= \eta_{1,m-s+1}(x_2; y_1, ..y_i, \frac{1}{s!}C'_{s+1}y, y_{i+s+1}, ..y_m) \\ &= \frac{1}{s!}\frac{1}{(m-s+1)!}C'_{s+1}C'_{m-s}y. \end{aligned}$$

As in the previous case, the second unshuffle of $x_2 \otimes x_2$ yields the same term with opposite sign.

Next consider the elements of the form $x_1 \otimes x_2 \otimes y^{\otimes m}$.

When $m = 1$ the OCHA relation has the form

$$\eta_{1,1}(\hat{l}_2(x_1, x_2); y) - \eta_{1,1}(x_1; \eta_{1,1}(x_2; y)) + \eta_{1,1}(x_2; \eta_{1,1}(x_1; y))$$
$$-\eta_{1,2}(x_1; \eta_{1,0}(x_2), y) - \eta_{1,2}(x_1; y, \eta_{1,0}(x_2)) + \eta_{1,2}(x_2; \eta_{1,0}(x_1), y) + \eta_{1,2}(x_2; y, \eta_{1,0}(x_1))$$

$= \hat{l}_2(\hat{l}_2(x_1, x_2), y) - \hat{l}_2(x_1, \hat{l}_2(x_2, y)) + \hat{l}_2(x_2, \hat{l}_2(x_1, y))$
$-\frac{1}{2}\hat{l}_3(x_1, \hat{l}_1(x_2), y) - \frac{1}{2}\hat{l}_3(x_1, y, \hat{l}_1(x_2)) + \frac{1}{2}\hat{l}_3(x_2, \hat{l}_1(x_1), y) + \frac{1}{2}\hat{l}_3(x_2, y, \hat{l}_1(x_1))$

$$= \hat{l}_2(x_1), y) - \hat{l}_2(x_1, 0) + \hat{l}_2(x_2, y) - \frac{1}{2}\hat{l}_3(x_1, y, y) - \frac{1}{2}\hat{l}_3(x_1, y, y) + \frac{1}{2}\hat{l}_3(x_2, y, y) + \frac{1}{2}\hat{l}_3(x_2, y, y)$$
$$= y - 0 + 0 - 0 - 0 - \frac{1}{2}y - \frac{1}{2}y = 0$$

For $m > 1$, the first sum in the OCHA relation has the form (with each $y_i = y$)

$$\eta_{1,m}(\hat{l}_2(x_1, x_2); y_1, \ldots, y_m) = \eta_{1,m}(x_1; y_1, \ldots, y_m) = \frac{1}{m!}\hat{l}_{m+1}(x_1, y_1, \ldots, y_m) = 0$$

so we consider only the second sum and evaluate the terms separately.

$$\eta_{2,m}(x_1, x_2; \eta_{0,m}(y_1, \ldots, y_m)) = \frac{1}{m!}\hat{l}_{m+2}(x_1, x_2, \frac{1}{m!}\hat{l}_m(y_1, \ldots, y_m))$$
$$= \frac{1}{m!}\hat{l}_{m+2}(x_1, x_2, 0) = 0$$

and

$$-\eta_{1,m+1}(x_1; y_1, \ldots, y_i, \eta_{1,0}(x_2), y_{i+1}, \ldots, y_m) = -\frac{1}{(m+1)!}\hat{l}_{m+2}(x_1; y_1, \ldots, y_i, \hat{l}_1(x_2), y_{i+1}, \ldots, y_m)$$
$$= -\frac{1}{(m+1)!}\hat{l}_{m+2}(x_1; y_1, \ldots, y_i, y, y_{i+1}, \ldots, y_m) = 0, \forall i.$$

Next we have

$$-\eta_{1,m}(x_1; y_1, \ldots, y_i, \eta_{1,1}(x_2; y_{i+1}), \ldots, y_m) = \frac{1}{m!}\hat{l}_{m+1}(x_1, y_1, \ldots, \hat{l}_2(x_2, y_{i+1}), \ldots, y_m) = 0, \forall i.$$

In general, we have

$$-\eta_{1,m+1}(x_1; y_1, \ldots, y_i, \eta_{1,s}(x_2; y_{i+1}, \ldots, y_{i+s}), y_{i+s+1}, \ldots, y_m)$$
$= -\frac{1}{(m+1)!}\hat{l}_{m+2}(x_1, y_1, \ldots, \frac{1}{s!}\hat{l}_{1+s}(x_2, y_{i+1}, \ldots, y_{i+s}), y_{i+s+1}, \ldots, y_m) = 0$
unless $i = 0$ and $s = m$ in which case we have

$$-\hat{l}_2(x_1, \frac{1}{m!}\hat{l}_{1+m}(x_2, y_1, \ldots, y_m)) = -\hat{l}_2(x_1, \frac{1}{m!}C'_{m+1}y) = -\frac{1}{m!}C'_{m+1}y. \quad (1)$$

We next compute the analogous terms for the other (1,1) unshuffle with the order of x_1 and x_2 interchanged:

$$\sum_{i=0}^{m} \eta_{1,m+1}(x_2; y_1, \ldots, y_i, \eta_{1,0}(x_1), \ldots, y_m) = \frac{1}{(m+1)!} \sum_{i=0}^{m} \hat{l}_{m+2}(x_2, y_1, \ldots, \hat{l}_1(x_1), \ldots, y_m)$$

$$= \frac{1}{(m+1)!} \sum_{i=0}^{m} \hat{l}_{m+2}(x_2, y, \ldots, y_i, \ldots, y) = \frac{m+1}{(m+1)!} C'_{m+2} y = \frac{1}{m!} C'_{m+2} y. \tag{2}$$

The final possibly non-zero terms occur in the sum

$$\sum_{i=0}^{m-1} \eta_{1,m}(x_2; y_1, \ldots, y_i, \eta_{1,1}(x_1; y_{i+1}), y_{i+2}, \ldots, y_m) = \frac{1}{m!} \sum_{i=0}^{m-1} \hat{l}_{m+1}(x_2, y_1, \ldots, y_i, \hat{l}_2(x_1, y_{i+1}), \ldots, y_m)$$

$$= \frac{1}{m!} \sum_{i=o}^{m-1} \hat{l}_{m+1}(x_2, y_1, \ldots, y_i, y, \ldots, y_m) = \frac{m}{m!} C'_{m+1} y. \tag{3}$$

We collect the coefficients $\frac{1}{m!}(-C'_{m+1} + C'_{m+2} + mC'_{m+1})$ from equations (1), (2), and (3) and evaluate:

$$\begin{aligned} \frac{1}{m!}(-C'_{m+1} + C'_{m+2} + mC'_{m+1}) &= \frac{1}{m!}(C'_{m+2} + (m-1)C'_{m+1}) \\ &= \frac{1}{m!}((-1)^{m+2}(m-1)! + (-1)^{m+1}(m-1)(m-2)!) \\ &= \frac{1}{m!}((-1)^{m+2}(m-1)! + (-1)^{m+1}(m-1)!) = 0. \end{aligned}$$

A final non-trivial case is that in which we consider the element of the form $x_1 \otimes x_2 \otimes x_2$. The only non-zero summand is

$$\eta_{2,0}(\hat{l}_2(x_1, x_2), x_2)$$

which occurs twice with opposite signs that result from the unshuffle permutations.

□

We say that this example is "small"in two senses. First of all, the underlying vector space is 3-dimensional. Secondly, the only non-trivial operations are the Lie bracket $\hat{l}_2 : W_{-1} \otimes W_{-1} \to W_{-1}$ and the maps $\eta_{1,q} : W_{-1} \otimes W_0^{\otimes q} \to W_0$ for $q \geq 0$. Even though we may impose a trivial A_∞ algebra structure on W_0, the presence of the non-trivial operation $\eta_{1,0}$ gives rise to an open-closed homotopy algebra structure rather than that of an A_∞ algebra over an L_∞ algebra. In string field theory the operation $\eta_{1,0}$ corresponds to the opening of a closed string into an open string.

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