

Strongly homotopy Lie algebras

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1. Introduction.

Strongly homotopy Lie algebras first made their appearance in a supporting role in deformation theory [11]. The philosophy that every deformation problem is directed by a differential graded Lie algebra leads, in the context of deformation theory of a differential graded algebra A , to a spectral sequence of which the E_2 -term is naturally a strongly homotopy Lie algebra.

For a topological space S , the homotopy groups $\pi_*(\Omega S)$ form a graded Lie algebra which can be extended non-trivially (though non-canonically) to a strongly homotopy Lie algebra which reflects more accurately the homotopy type of S . The relevant operations represent the higher order Whitehead products on S . In the stable range, the basic products are given by composition and higher order composition products; more details are given in [12]).

More recently, closed string field theory, especially in the hands of Zwiebach and his collaborators, [15], [14] has produced a particular strongly homotopy Lie Algebra. Lada and Stasheff [6] provided an exposition of the basic ingredients of the theory of strongly homotopy Lie algebras sufficient for the underpinnings of the physically relevant examples. That work left open several questions naturally suggested by comparison with the theory of differential graded Lie algebras. The present paper addresses such questions in characteristic zero and is complementary to what currently exists in the literature, both physical and mathematical.

Both strongly homotopy Lie algebras and strongly homotopy associative algebras can be expressed in terms of n -ary operations, respectively l_n and μ_n for all natural numbers $n \geq 1$. The defining relations when restricted to situations with $n \leq m$ yield corresponding structures called $L(m)$ and $A(m)$ algebras respectively. Section 2 of this work contains the basic definitions and notation regarding $L(m)$ structures and is highlighted by the expected correspondence between an $L(m)$ structure on a differential graded vector space L , and a degree -1 coderivation that is a

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differential on the cofree cocommutative coalgebra generated by the suspension of L ; this is the content of Theorem 2.3.

In Section 3 we demonstrate the "strong homotopy" analog of the usual relation between Lie and associative algebras. Theorem 3.1 implies that appropriate "skew-symmetrization" of a strongly homotopy associative algebra is a strongly homotopy Lie algebra. Here, the condition that the field has characteristic zero is essential. Theorem 3.3 completes the equivalence of homotopy categories: there is a functor "universal enveloping strongly homotopy associative algebra" from $L(m)$ -algebras to $A(m)$ -algebras which is left adjoint to the "higher order commutators" functor.

Properties of this universal enveloping $A(m)$ functor are studied in Section 4. In particular, there exists a strict symmetric monoidal structure on the category of unital $A(m)$ algebras such that the universal enveloping $A(m)$ algebra functor carries a natural structure of a unital coassociative cocommutative coalgebra with respect to this monoidal structure. Propositions 4.1 and 4.3 contain the details of these properties.

Section 5 is concerned with $L(m)$ -modules and introduces a notion of a weak homotopy map from an $L(m)$ -algebra to a differential graded Lie algebra; this generalizes certain maps considered by Retakh [10]. A relationship between such maps and such modules is given in Theorem 5.3.

After this paper was written, we learned of work of Hanlon and Wachs [5] on Lie_k -algebras. Developed independently, these turn out to be special cases of $L(k)$ -algebras in which only the 'last' map is non-zero.

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2. Basic definitions and notations.

All algebraic objects in the paper will be considered over a fixed field \mathbf{k} of characteristic zero. We will systematically use the Koszul sign convention meaning that whenever we commute two "things" of degrees p and q , respectively, we multiply the sign by $(-1)^{pq}$. Our conventions concerning graded vector spaces, permutations, shuffles, etc., will follow closely those of [8].

For graded indeterminates x_1, \dots, x_n and a permutation $\sigma \in \mathcal{S}_n$ define the *Koszul sign* $\epsilon(\sigma) = \epsilon(\sigma; x_1, \dots, x_n)$ by

$$x_1 \wedge \dots \wedge x_n = \epsilon(\sigma; x_1, \dots, x_n) \cdot x_{\sigma(1)} \wedge \dots \wedge x_{\sigma(n)},$$

which has to be satisfied in the free graded commutative algebra $\wedge(x_1, \dots, x_n)$. Define also

$\chi(\sigma) = \chi(\sigma; x_1, \dots, x_n) := \text{sgn}(\sigma) \cdot \epsilon(\sigma; x_1, \dots, x_n)$. We say that $\sigma \in \mathcal{S}_n$ is an $(j, n-j)$ -unshuffle, $0 \leq j \leq n$, if $\sigma(1) < \dots < \sigma(j)$ and $\sigma(j+1) < \dots < \sigma(n)$.

Definition 2.1. An $L(m)$ -structure on a graded vector space L is a system $\{l_k \mid 1 \leq k \leq m, k < \infty\}$ of linear maps $l_k : \otimes^k L \rightarrow L$ with $\deg(l_k) = k - 2$ which are antisymmetric in the sense that

$$(1) \quad l_k(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = \chi(\sigma) l_k(x_1, \dots, x_n)$$

for all $\sigma \in \mathcal{S}_n$ and $x_1, \dots, x_n \in L$, and, moreover, the following generalized form of the Jacobi identity is supposed to be satisfied for any $n \leq m$:

$$(2) \quad \sum_{i+j=n+1} \sum_{\sigma} \chi(\sigma) (-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0,$$

where the summation is taken over all $(i, n-i)$ -unshuffles with $i \geq 1$.

Example 2.2. An $L(1)$ -algebra structure on L consists of a degree -1 endomorphism l_1 and the Jacobi identity (2) reduces to $l_1^2 = 0$, i.e. an $L(1)$ -algebra is just a differential space.

An $L(2)$ -algebra has one more operation, a bilinear map l_2 which we denote more suggestively as $[-, -]$. The antisymmetry condition (1) gives $[x, y] = -(-1)^{|x||y|}[y, x]$ and the Jacobi condition (2) says that

$$l_1([x, y]) = [l_1(x), y] + (-1)^{|x|}[x, l_1(y)],$$

in other words, an $L(2)$ -algebra is just an antisymmetric nonassociative nonunital differential graded algebra.

For an $L(3)$ -algebra we have again one more antisymmetric operation, l_3 , which is the contracting homotopy for the classical Jacobi identity:

$$\begin{aligned} & (-1)^{|x||z|}[[x, y], z] + (-1)^{|y||z|}[[z, x], y] + (-1)^{|x||y|}[[y, z], x] = \\ & (-1)^{|x||z|+1} \cdot \left\{ l_1 l_3(x, y, z) + l_3(l_1(x), y, z) + (-1)^{|x|} \cdot l_3(x, l_1(y), z) + (-1)^{|x|+|y|} \cdot l_3(x, y, l_1(z)) \right\}. \end{aligned}$$

$L(\infty)$ -algebras are sometimes, especially in the physical literature, called also *homotopy Lie algebras*, but one must beware that this name has already been reserved for $\pi_*(\Omega S)$, the homotopy algebra of the loop space of a topological space S with the graded Lie algebra structure induced by the Samelson product. So, it is more appropriate to call them *strongly* homotopy Lie algebras or *sh Lie* algebras, as in [6]

Let $L = (L, l_k)$ and $L' = (L', l'_k)$ be two $L(m)$ -algebras. By a *map* of L to L' we mean a linear degree zero map $g : L \rightarrow L'$ which commutes with the structure maps in the sense that

$$g \circ l_k = l'_k \circ g^{\otimes k}, \quad 1 \leq k \leq m.$$

Denote by $\mathbf{L}(m)$ the category of $L(m)$ -algebras and their homomorphisms in the above sense. $\mathbf{L}(m)$ is an equationally given algebraic category, a fact which we use in the next paragraph.

Let \mathbf{Vect} be the category of graded vector spaces. Denote by $\mathbf{Vect}^p(V, W)$ the set of linear homogeneous maps $f : V \rightarrow W$ of degree p . For $V \in \mathbf{Vect}$, let $\uparrow V$ (resp. $\downarrow V$) be the suspension (resp. the desuspension) of V , i.e. the graded vector space defined by $(\uparrow V)_p = V_{p-1}$ (resp. $(\downarrow V)_p = V_{p+1}$). By $\#V$ we denote the dual of V , i.e. the graded vector space $(\#V)_p := \mathbf{Vect}^p(V, \mathbf{k}) = \text{Lin}(V_{-p}, \mathbf{k})$, the space of linear maps from V_p to \mathbf{k} . For a graded vector space V we have the natural map $\uparrow : V \rightarrow \uparrow V$; let $\uparrow^{\otimes n}$ denote $\otimes^n \uparrow : \otimes^n V \rightarrow \otimes^n \uparrow V$, the meaning of $\downarrow^{\otimes n}$ being analogous. Notice that $\uparrow^{\otimes n} \circ \downarrow^{\otimes n} = \downarrow^{\otimes n} \circ \uparrow^{\otimes n} = (-1)^{\frac{n(n-1)}{2}} \cdot \mathbb{1}$, as a side effect of the Koszul sign convention.

For a graded vector space V , $\wedge V$ will denote the free graded commutative algebra on V . As usual, by $\wedge^n V$ we mean the subspace of $\wedge V$ consisting of elements of length n , the notations like $\wedge^{\leq n} V$ having the obvious meaning. We will need also the dual analog of this object. Namely, for a graded vector space W , consider the coalgebra ${}^c\wedge W$ which, as a vector space, coincides with $\wedge W$, but the comultiplication Δ is given by $\Delta = \mathbb{1} \otimes 1 + \overline{\Delta} + 1 \otimes \mathbb{1}$, where the reduced diagonal $\overline{\Delta}$ is defined to be

$$\overline{\Delta}(w_1 \wedge \cdots \wedge w_n) = \sum_{1 \leq j \leq n-1} \sum_{\sigma} \epsilon(\sigma) (w_{\sigma(1)} \wedge \cdots \wedge w_{\sigma(j)}) \otimes (w_{\sigma(j+1)} \wedge \cdots \wedge w_{\sigma(n)}),$$

where σ runs through all $(j, n-j)$ unshuffles. ${}^c\wedge W$ is clearly a cocommutative (coassociative, counital) connected coalgebra. It has the universal property, dual to the universal property characterizing the freeness of $\wedge V$ but, as usual in the co-algebraic world, not exactly.

To describe the universal property, introduce, for a given (counital) coalgebra $C = (C, \Delta)$, the filtration $\{F_i C\}_{i \geq 0}$ inductively by $F_0 := 0$ and $F_i C := \{c \in C \mid \overline{\Delta}(c) \in F_{i-1} C \otimes F_{i-1} C\}$, $i \geq 1$. We say that C is *connected* if $C = \bigcup F_i C$. Notice that ${}^c\wedge W$ itself is connected, with $F_i {}^c\wedge W = {}^c\wedge^{\leq i} W$, the subspace corresponding to $\wedge^{\leq i} W$ under the identification ${}^c\wedge W = \wedge W$ of graded vector spaces.

Let $\pi : {}^c\wedge W \rightarrow W$ be the natural projection. The universal property of ${}^c\wedge W$ then says that, for any cocommutative connected coalgebra C and for any linear map $\psi : C \rightarrow W$, there exists

exactly one coalgebra homomorphism $g : C \rightarrow {}^c\wedge W$ such that the diagram

$$\begin{array}{ccc}
 C & \xrightarrow{g} & {}^c\wedge W \\
 & \searrow \psi & \downarrow \pi \\
 & & W
 \end{array}$$

commutes.

Denote by $\iota_m : \wedge^{\leq m} W \hookrightarrow {}^c\wedge W$ the obvious inclusion and, dually, let $\pi_m : \wedge V \rightarrow \wedge^{\leq m} V$ be the natural projection.

Theorem 2.3. *There is one-to-one correspondence between $L(m)$ -algebra structures on a graded vector space L and degree -1 coderivations δ on the coalgebra ${}^c\wedge W$, $W := \uparrow L$, with the property that $\delta^2 \circ \iota_m = 0$.*

If the space L is of finite type, then $L(m)$ algebra structure on L can be described also by a degree -1 derivation d on $\wedge V$, $V = \downarrow \#L$, with the property that $\pi_m \circ d^2 = 0$.

The first part of the theorem was, for $m = \infty$, proved in [6]. The second part is, for $m = \infty$, a folk-lore result and it is related to the fact that the Koszul dual of the category of graded Lie algebras is the category of graded commutative algebras [1]. The case of a general m is an easy generalization, following the lines of the proof of the similar statement for $A(m)$ -algebras, see [8, Example 1.9]. We do not aim to give a proof here, but the explicit description of the correspondences will be useful in the sequel.

First, recall that, for a bimodule N over $\wedge V$, the space $\text{Der}^p(\wedge V, N)$ of degree p derivations of the algebra $\wedge V$ in the bimodule N has a very easy description:

$$(3) \quad \text{Der}^p(\wedge V, N) \cong \mathbf{Vect}^p(V, N).$$

The dual statement for the coalgebra ${}^c\wedge W$ and a bicomodule M over ${}^c\wedge W$ needs the assumption that the comodule M is *connected* meaning that, by definition, ${}^c\wedge W \oplus M$ with the obvious coalgebra structure is connected. Observe that ${}^c\wedge W$ is a connected comodule over itself. We have the following statement.

Lemma 2.4. *For a connected bicomodule N over the coalgebra ${}^c\wedge W$ we have an isomorphism*

$$\text{Coder}^p(M, {}^c\wedge W) \cong \mathbf{Vect}^p(M, W),$$

induced by the correspondence $\theta \mapsto \pi \circ \theta$, $\pi : {}^c\wedge W \rightarrow W$ being the projection.

Proof. Observe first that $\downarrow^p M$ has a natural cobimodule structure and that $\text{Coder}^p(M, {}^c\wedge W) \cong \text{Coder}^0(\downarrow^p M, {}^c\wedge W)$. Thus we can reduce the statement of the lemma to the case $p = 0$.

Let $\text{Coalg}(-, -)$ stand for the set of coalgebra maps and consider the map

$$(4) \quad \text{Coalg}({}^c\wedge W \oplus M, {}^c\wedge W) \longrightarrow \text{Coalg}({}^c\wedge W, {}^c\wedge W)$$

given by the restriction on ${}^c\wedge W$. Then $\text{Coder}^0(M, {}^c\wedge W)$ obviously consists of those elements of $\text{Coalg}({}^c\wedge W \oplus M, {}^c\wedge W)$ which restrict to the identity in $\text{Coalg}({}^c\wedge W, {}^c\wedge W)$. Using the universal property of ${}^c\wedge W$, the map of (4) can be described also as

$$\mathbf{Vect}({}^c\wedge W \oplus M, W) \cong \mathbf{Vect}({}^c\wedge W, M) \oplus \mathbf{Vect}(M, W) \xrightarrow{\text{proj.}} \mathbf{Vect}({}^c\wedge W, W)$$

and the Lemma immediately follows. ■

Suppose that $\{l_k \mid 1 \leq k \leq m\}$ is an $L(m)$ -structure on a graded vector space L as in Definition 2.1. Let $W := \uparrow L$ and define degree -1 linear maps $\bar{\delta}_k : \otimes^k W \rightarrow W$ by $\bar{\delta}_k := (-1)^{\frac{k(k-1)}{2}} \cdot \uparrow \circ l_k \circ \downarrow^{\otimes k}$, $1 \leq k \leq m$. Then, by the antisymmetry property (1) of the maps l_k , the maps $\bar{\delta}_k$ are symmetric in the sense that

$$\bar{\delta}_k(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = \epsilon(\sigma) \bar{\delta}_k(x_1, \dots, x_k), \quad \sigma \in \mathcal{S}_k,$$

which means that they factor to the maps (denoted by the same symbol) $\bar{\delta}_k : \wedge^k W \rightarrow W$. By Lemma 2.4 there exists exactly one coderivation $\delta \in \text{Coder}^{-1}({}^c\wedge W)$ (= an abbreviation for $\text{Coder}^{-1}({}^c\wedge W, {}^c\wedge W)$) with the property that

$$\pi \circ \delta = \begin{cases} \bar{\delta}_k(w), & \text{for } 1 \leq k \leq m, \\ 0, & \text{otherwise.} \end{cases}$$

The ‘‘Jacobi identity’’ (2) is then equivalent to $\delta^2 \circ \iota_m = 0$.

On the other hand, the maps l_k can be reconstructed from δ as $l_k = \downarrow \circ \bar{\delta}_k \circ \uparrow^{\otimes k}$ with $\bar{\delta}_k$ defined as the composition

$$\otimes^k W \xrightarrow{\text{proj.}} \wedge^k W \hookrightarrow {}^c\wedge W \xrightarrow{\delta} {}^c\wedge W \xrightarrow{\pi} W.$$

This gives the correspondence of the first part of the theorem. The description of the second one is similar. Let $V = \downarrow \#L$ and define $\bar{d}_k : \wedge^k V \rightarrow V$ as the composition

$$V \xrightarrow{\uparrow} \#L \xrightarrow{\#l_k} \# \otimes^k L \xrightarrow{\downarrow^{\otimes k}} \# \otimes^k V \xrightarrow{\text{proj.}} \wedge^k V,$$

multiplied by $(-1)^{\frac{k(k-1)}{2}}$, for $k \leq m$, and let $\bar{d}_k := 0$ otherwise. By (3) it defines a derivation $d \in \text{Der}^{-1}(\wedge V)$ (= an abbreviation for $\text{Der}^{-1}(\wedge V, \wedge V)$). The Jacobi identity (2) is then equivalent to $d^2 = 0$.

On the other hand, starting from d , define \bar{d}_k as the composition

$$V \xrightarrow{d} \wedge V \xrightarrow{\text{proj.}} \wedge^k V \xrightarrow{\text{incl.}} \otimes^k V.$$

Then we can reconstruct l_k 's as $l_k = \#(\downarrow \circ \bar{d}_k \circ \uparrow^{\otimes k})$.

3. Symmetrization.

The usual relationship between Lie algebras and associative algebras carries over directly to this homotopy setting. Recall [13, p. 294] that an $A(m)$ structure on a graded vector space V is a collection $\{\mu_k | 1 \leq k \leq m\}$ of linear maps $\mu_k : \otimes^k V \rightarrow V$ with the degree of μ_k equal to $k - 2$. These maps are required to satisfy the identity

$$\sum_{\lambda=0}^{n-1} \sum_{k=1}^{n-\lambda} (-1)^{k+\lambda+k\lambda+nk+k(|a_1|+\dots+|a_\lambda|)} \mu_{n-k+1}(a_1, \dots, a_\lambda, m_k(a_{\lambda+1}, \dots, a_{\lambda+k}), a_{\lambda+k+1}, \dots, a_n) = 0.$$

We note that μ_1 is a differential for V , μ_2 is a multiplication, and the μ_k 's are higher associating homotopies.

A homomorphism ($A(m)$ -map) between two $A(m)$ -algebras (V, μ_i) and (V', μ'_i) is a linear map $f : V \rightarrow V'$ of degree 0 such that

$$f \circ \mu_n = \mu'_n \circ f^{\otimes n}, n = 1, \dots, m.$$

We denote by $\mathbf{A}(m)$ the category of $A(m)$ -algebras and $A(m)$ -maps. See [8, Example 1.9] for a thorough discussion.

We also recall that an $A(m)$ structure on a graded vector space V may be described by a degree -1 coderivation $\partial : {}^cTW \rightarrow {}^cTW$ with $\partial^2 = 0$. Here, cTW is the coassociative coalgebra with underlying vector space $TW = \bigoplus_k \otimes^k W$ and with the reduced diagonal $\bar{\Delta}$ given by

$$\bar{\Delta}(w_1 \otimes \dots \otimes w_n) = \sum_{i=1}^{n-1} (w_1 \otimes \dots \otimes w_i) \otimes (w_{i+1} \otimes \dots \otimes w_n),$$

$W = \uparrow V$. Let $\pi' : {}^cTW \rightarrow W$ denote the natural projection. The $A(m)$ analog of Theorem 1.3 gives us that the $A(m)$ structure maps μ_k can be recovered from ∂ by $\mu_k = \downarrow \bar{\partial}_k \uparrow^{\otimes k}$ where $\bar{\partial}_k$ is the composition

$$\otimes^k W \xrightarrow{\text{incl.}} {}^cTW \xrightarrow{\partial} {}^cTW \xrightarrow{\pi'} W.$$

Details may be found in [2].

Theorem 3.1. *An $\mathbf{A}(m)$ -structure $\{\mu_n : \otimes^n V \longrightarrow V\}$ on the graded vector space V induces an $\mathbf{L}(m)$ -structure $\{l_n : \otimes^n V \longrightarrow V\}$ where*

$$l_n(v_1 \otimes \dots \otimes v_n) := \sum_{\sigma \in \mathcal{S}_n} \chi(\sigma) \mu_n(v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}), \quad 1 \leq n \leq m.$$

This correspondence defines a functor $(-)_L : \mathbf{A}(m) \longrightarrow \mathbf{L}(m)$.

Proof. Consider the injective coalgebra map $S : {}^c\Lambda W \longrightarrow {}^cTW$ given by

$$S(w_1 \wedge \dots \wedge w_n) = \sum_{\sigma \in \mathcal{S}_n} \epsilon(\sigma) (w_{\sigma(1)} \otimes \dots \otimes w_{\sigma(n)}).$$

Using Lemma 1.4, we extend the linear map $\pi' \partial S : {}^c\Lambda W \longrightarrow W$ to the unique coderivation $\delta : {}^c\Lambda W \longrightarrow {}^c\Lambda W$ which has the property that $\pi \delta = \pi' \partial S$. Since $\delta^2 : {}^c\Lambda W \longrightarrow {}^c\Lambda W$ is a coderivation, to show that $\delta^2 = 0$ we need only show that $\pi \delta^2 = 0$. But, $\pi \delta^2 = \pi' \partial S \delta$ which is equal to 0 if $S \delta = \partial S$. Since S is a coalgebra map, $S \delta$ and $\partial S \in \text{Coder}({}^c\Lambda W, {}^cTW)$ and so we need examine only $\pi' S \delta$ and $\pi' \partial S$; since $\pi' S = \pi$, $\pi' S \delta = \pi \delta$ whereas $\pi' \partial S = \pi \delta$ by definition. The resulting $\mathbf{L}(m)$ -structure on V now follows from Theorem 1.3. In addition, if $f : (V, \mu_n) \longrightarrow (V', \mu'_n)$ is an $\mathbf{A}(m)$ -map so that $\mu'_n \circ f^{\otimes n} = f \circ \mu_n$, then $f \circ l_n(v_1 \otimes \dots \otimes v_n) = \sum_{\sigma \in \mathcal{S}_n} \chi(\sigma) f \circ \mu_n(v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}) = \sum_{\sigma \in \mathcal{S}_n} \chi(\sigma) \mu'_n \circ f^{\otimes n}(v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}) = l'_n \circ f^{\otimes n}(v_1 \otimes \dots \otimes v_n)$ which shows the functoriality of our construction. ■

Remark 3.2. It should be clear that for $n = 2$, $l_2(v_1 \otimes v_2) = \mu_2(v_1 \otimes v_2) - (-1)^{|v_1||v_2|} \mu_2(v_2 \otimes v_1)$ is the usual graded commutator. For $n > 2$, the l_n 's are the appropriate symmetrization of the associating homotopies.

If $A = (A, \partial, \cdot)$ is an associative differential graded algebra considered in an obvious way as an $\mathbf{A}(m)$ -algebra for some $m \geq 2$ (see [8, Example 1.5]), then A_L is the usual commutator Lie algebra associated to A .

The following proposition follows from the fact that $(-)_L : \mathbf{A}(m) \rightarrow \mathbf{L}(m)$ is an algebraic functor and thus has a left adjoint.

Theorem 3.3. *There is a functor $\mathcal{U}_m : \mathbf{L}(m) \longrightarrow \mathbf{A}(m)$ that is left adjoint to $(-)_L$. \mathcal{U}_m is called the universal enveloping $\mathbf{A}(m)$ -algebra functor for $\mathbf{L}(m)$ -algebras.*

There exists another construction of a ‘‘universal enveloping algebra’’ which gives, for any sh Lie algebra L , an *associative* (not $\mathbf{A}(\infty)$) algebra characterized by a certain universal property

with respect to L -modules, see [3]. We used the name “universal enveloping $A(m)$ -algebra” instead of just “universal enveloping algebra” to distinguish between these two constructions.

There is a description of $\mathcal{U}_m(L)$ that is analogous to the classical description of the universal enveloping algebra of a Lie algebra. We begin with a graded vector space L with its $L(m)$ -structure $\{l_n\}$. Let $\mathcal{F}_m(L)$ be the free $A(m)$ -algebra generated by the vector space L with $A(m)$ -structure maps denoted by $\{\mu_n\}$. Let I denote the ideal in $\mathcal{F}_m(L)$ generated by the relations

$$\sum_{\sigma \in S_n} \chi(\sigma) \mu_n(\xi_{\sigma(1)}, \dots, \xi_{\sigma(n)}) = l_n(\xi_1, \dots, \xi_n)$$

where $\xi_1, \dots, \xi_n \in L$. Let $\mathcal{U}_m(L) = \mathcal{F}_m(L)/I$ and $j : L \rightarrow \mathcal{U}_m(L)$ be the natural inclusion. $\mathcal{U}_m(L)$ is then universal in the following sense: given a linear map $f : L \rightarrow A$ where A is an $A(m)$ -algebra such that $f : L \rightarrow A_L$ is an $L(m)$ -homomorphism, there is a unique $A(m)$ -map $\hat{f} : \mathcal{U}_m(L) \rightarrow A$ such that $\hat{f} \circ j = f$. To see this, note that there is a unique homomorphism of $A(m)$ -algebras, $\hat{f} : \mathcal{F}_m(L) \rightarrow A$ such that $\hat{f} \circ j = f$ since $\mathcal{F}_m(L)$ is free. We need only check that $\hat{f}(I) = 0$. We denote the $A(m)$ -structure on A by $\{\hat{\mu}_n\}$ and its corresponding commutator $L(m)$ -structure by $\{\hat{l}_n\}$.

We apply \hat{f} to each side of the equation that defines the ideal I and obtain

$$\begin{aligned} \hat{f}\left(\sum_{\sigma} \chi(\sigma) \mu_n(\xi_{\sigma(1)}, \dots, \xi_{\sigma(n)})\right) &= \sum_{\sigma} \chi(\sigma) \hat{f} \mu_n(\xi_{\sigma(1)}, \dots, \xi_{\sigma(n)}) = \\ &= \sum_{\sigma} \chi(\sigma) \mu_n \hat{f}(\xi_{\sigma(1)}, \dots, \xi_{\sigma(n)}) = \sum_{\sigma} \chi(\sigma) \mu_n(f \xi_{\sigma(1)}, \dots, f \xi_{\sigma(n)}) \end{aligned}$$

on the left. On the other side we have

$$\hat{f}(l_n(\xi_1, \dots, \xi_n)) = f(l_n(\xi_1, \dots, \xi_n)) = \hat{l}_n(f \xi_1, \dots, f \xi_n) = \sum_{\sigma} \chi(\sigma) \mu_n(f \xi_{\sigma(1)}, \dots, f \xi_{\sigma(n)}).$$

This shows that $\hat{f}(I) = 0$, therefore \hat{f} factors to the requisite map $\mathcal{U}_m(L) \rightarrow A$.

4. Some properties of $\mathcal{U}_m(L)$.

The aim of this section is to show the existence of a strict symmetric monoidal structure on the category $\mathbf{A}(m)$ of unital $A(m)$ -algebras such that the universal enveloping $A(m)$ -algebra constructed in the previous section carries a natural structure of a unital coassociative cocommutative coalgebra with respect to this monoidal structure, c.f. the classical analog of this result [3].

Let A and B be two $A(m)$ -algebras. Choose free presentations $A = \mathcal{F}_m(X_A)/(R_A)$ and $B = \mathcal{F}_m(X_B)/(R_B)$. Then define

$$(5) \quad A \square B := \mathcal{F}_m(X_A \oplus X_B)/(R_A, S_{A,B}, R_B),$$

where $S_{A,B}$ is the ideal generated by the relations

$$\sum_{\sigma \in S_n} \chi(\sigma; x_1, \dots, x_n) \cdot \mu_n(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = 0,$$

with $x_{i_1}, \dots, x_{i_s} \in X_A$, $x_{j_1}, \dots, x_{j_t} \in X_B$, where $\{i_1, \dots, i_s, j_1, \dots, j_t\}$ is a decomposition of $\{1, \dots, n\}$, $s, t \geq 1$, and $s + t = n$.

In the following proposition, $\mathbf{1}$ will denote the trivial unital $\mathbf{A}(m)$ -algebra, $\mathbf{1} = (\mathbf{k}, \mu_i)$, with $\mu_i(1 \otimes \dots \otimes 1) = 1$ for $i = 2$ and $\mu_i(1 \otimes \dots \otimes 1) = 0$ otherwise.

Proposition 4.1. *The operation \square introduced above induces on the category $\mathbf{A}(m)$ the structure of a strict symmetric monoidal category with $\mathbf{1}$ as the unit object.*

Proof. Notice that the formula (5) gives a well-defined functor $\square : \mathbf{A}(m) \times \mathbf{A}(m) \rightarrow \mathbf{A}(m)$. The obvious fact that $S_{A,B} = S_{B,A}$ gives the symmetry $s : A \square B \rightarrow B \square A$.

Let $C = \mathcal{F}_m(X_C)/(R_C)$ be a third $\mathbf{A}(m)$ -algebra. We have, by definition,

$$\begin{aligned} A \square (B \square C) &= \mathcal{F}_m(X_A \oplus X_B \oplus X_C)/(R_A, R_B, R_C, S_{A,B \oplus C}, S_{B,C}), \\ (A \square B) \square C &= \mathcal{F}_m(X_A \oplus X_B \oplus X_C)/(R_A, R_B, R_C, S_{A,B}, S_{A \oplus B, C}). \end{aligned}$$

On the other hand, clearly

$$(S_{A,B \oplus C}, S_{B,C}) = (S_{A,B}, S_{A \oplus B, C}) = (S_{A,B}, S_{B,C}, S_{C,A})$$

which easily gives the ‘‘associativity isomorphism’’ $\alpha_{A,B,C} : A \square (B \square C) \rightarrow (A \square B) \square C$.

Finally, if $\mathbf{0}$ denotes the trivial vector space, then $\mathbf{1} = \mathcal{F}_m(\mathbf{0})/(\mathbf{0})$ and we see immediately that $A \square \mathbf{1} = \mathbf{1} \square A = A$. The reader may easily verify that the structures constructed above satisfy the axioms of a strict symmetric monoidal category as they are listed, for example, in [9]. ■

Let $L' = (L', l'_i)$ and $L'' = (L'', l''_j)$ be two $L(m)$ -algebras. Define their *direct product* $L' \times L'' = (L' \oplus L'', l_n)$ by

$$l_n(\xi_1, \dots, \xi_n) := \begin{cases} l'_n(\xi_1, \dots, \xi_n), & \text{if all } \xi_i \in L', \\ l''_n(\xi_1, \dots, \xi_n), & \text{if all } \xi_i \in L'', \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 4.2. *For any two $L(m)$ -algebras L' and L'' there is a natural isomorphism*

$$\mathcal{U}_m(L' \oplus L'') \cong \mathcal{U}_m(L') \square \mathcal{U}_m(L'')$$

of $\mathbf{A}(m)$ -algebras.

Proof. Using the description of the universal enveloping $\mathbf{A}(m)$ -algebra as it is given at the end of the previous paragraph, we have

$$\mathcal{U}_m(L' \times L'') = \mathcal{F}_m(L' \oplus L'')/(I),$$

with I is the ideal generated by the relations

$$(6) \quad \sum_{\sigma \in S_n} \chi(\sigma) \cdot \mu_n(\xi_{\sigma(1)}, \dots, \xi_{\sigma(n)}) = l_n(\xi_1, \dots, \xi_n),$$

where $\xi_1, \dots, \xi_n \in L' \oplus L''$. It is immediate to see that $I = (R', S, R'')$, where R' (resp. R'') is generated by the relations (6) with $\xi_i \in L'$ (resp. $\xi_i \in L''$) and $S = S_{L', L''}$. The proposition now follows from the definition of the \square -product. \blacksquare

Let L be an $\mathbf{L}(m)$ -algebra and let $\delta : L \rightarrow L \times L$ be the homomorphism given by $\delta(\xi) := \xi \oplus \xi$. This map induces, by the functoriality of $\mathcal{U}_m(-)$, the $\mathbf{A}(m)$ -map $\Delta : \mathcal{U}_m(L) \rightarrow \mathcal{U}_m(L \times L) = \mathcal{U}_m(L) \square \mathcal{U}_m(L)$.

Proposition 4.3. *The homomorphism $\Delta : \mathcal{U}_m(L) \rightarrow \mathcal{U}_m(L) \square \mathcal{U}_m(L)$ induces on $\mathcal{U}_m(L)$ the structure of a cocommutative coassociative coalgebra in the monoidal category $(\mathbf{A}(m), \square, \mathbf{1})$, the counit given by the augmentation $\epsilon : \mathcal{U}_m(L) \rightarrow \mathbf{1}$.*

Proof. The coassociativity of Δ means that the diagram

$$\begin{array}{ccc} \mathcal{U}_m(L) & \xrightarrow{\Delta} & \mathcal{U}_m(L) \square \mathcal{U}_m(L) \\ \Delta \downarrow & & \downarrow \Delta \square \mathbb{1} \\ \mathcal{U}_m(L) \square \mathcal{U}_m(L) & \xrightarrow{\mathbb{1} \square \Delta} & \mathcal{U}_m(L) \square \mathcal{U}_m(L) \square \mathcal{U}_m(L) \end{array}$$

commutes. But this diagram is obtained by applying the functor $\mathcal{U}_m(-)$ on the diagram

$$\begin{array}{ccc} L & \xrightarrow{\delta} & L \times L \\ \delta \downarrow & & \downarrow \delta \times \mathbb{1} \\ L \times L & \xrightarrow{\mathbb{1} \times \delta} & L \times L \times L \end{array}$$

which obviously commutes.

The fact that ϵ is a left counit means, by definition, that the composition

$$\mathcal{U}_m(L) \xrightarrow{\Delta} \mathcal{U}_m(L) \square \mathcal{U}_m(L) \xrightarrow{\epsilon \square \mathbf{1}} \mathbf{1} \square \mathcal{U}_m(L) \xrightarrow{\cong} \mathcal{U}_m(L)$$

is the identity map. But this composition is obtained by applying the functor $\mathcal{U}_m(-)$ on the composition

$$L \xrightarrow{\delta} L \times L \xrightarrow{0 \times \mathbf{1}} \mathbf{0} \times L \xrightarrow{\cong} L$$

which is plainly the identity. The proof that ϵ is also a right counit is the same. ■

Remark 4.4. The arguments of this section seem to suggest a general scheme valid for all universal-enveloping-algebra-like functors $\mathcal{U} : \mathcal{A} \rightarrow \mathcal{B}$ (we know three examples of such functors, the “classical” universal enveloping algebra functor for Lie algebras [3], the universal enveloping algebra functor for Leibniz algebras of [7] and, of course, our functor $\mathcal{U}_m(-)$).

To construct a coassociative coalgebra structure on $\mathcal{U}(L)$, look for a strict monoidal structure $- \odot -$ on the category \mathcal{B} having the property that the functors $\mathcal{U}(- \times -)$ and $\mathcal{U}(-) \odot \mathcal{U}(-)$ ($- \times -$ denoting the direct product in the category \mathcal{A}) are naturally equivalent. Then the coassociative comultiplication on $\mathcal{U}(L)$ is induced by the diagonal map $\delta : L \times L \rightarrow L$. Examples of these suitable monoidal structures are: the tensor product $- \otimes -$ for the classical universal enveloping algebra functor, the free product for the universal enveloping algebra functor for Leibniz algebras and, of course, the operation $- \square -$ for our functor $\mathcal{U}_m(-)$.

5. $L(m)$ -modules.

We introduce two concepts in this section. The first is that of left modules over $L(m)$ -algebras. The second idea involves homomorphisms from $L(m)$ -algebras to differential graded Lie algebras. It is not surprising that these two ideas are closely related. We begin with the following

Definition 5.1. *Let $L = (L, l_i)$, be an $L(m)$ -algebra, and let M be a differential graded vector space with differential denoted by k_1 . Then a left L -module structure on M is a collection $\{k_n | 1 \leq n \leq m, n < \infty\}$ of linear maps of degree $n - 2$,*

$$k_n : \bigotimes^{n-1} L \otimes M \longrightarrow M,$$

such that

$$(7) \quad \sum_{i+j=n+1} \sum_{\sigma} \chi(\sigma) (-1)^{i(j-1)} k_j(k_i(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i)}), \xi_{\sigma(i+1)}, \dots, \xi_{\sigma(n)}) = 0$$

where σ ranges over all $(i, n - i)$ unshuffles, $\xi_1, \dots, \xi_{n-1} \in L$ and $\xi_n \in M$.

Several comments are in order. We assume that $\xi_n \in M$ while the other ξ_i 's $\in L$, and then according to the definition of $(i, n - i)$ unshuffles, it follows that either $\xi_{\sigma(i)} = \xi_n$ or $\xi_{\sigma(n)} = \xi_n$. In the first case then, we define

$$k_j(k_i(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i)}), \xi_{\sigma(i+1)}, \dots, \xi_{\sigma(n)}) := \alpha \cdot k_j(\xi_{\sigma(i+1)}, \dots, \xi_{\sigma(n)}, k_i(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i)}))$$

where

$$\alpha = (-1)^{j-1} \cdot (-1)^{(i+\sum_{k=1}^i |\xi_{\sigma(k)}|) \cdot (\sum_{k=i+1}^n |\xi_{\sigma(k)}|)}$$

according to the Koszul sign convention. In the second case, i.e. when $\xi_{\sigma(i)} \in L$, we take $k_i = l_i$.

It is not difficult to show that L -modules in the above sense are abelian group objects in the slice category $\mathbf{L}(m)/L$ of $L(m)$ -algebras over L .

Of course, the fundamental example of such a structure occurs in the situation when $M = L$ and each $k_i = l_i$, i.e., L is an $L(m)$ -module over itself. Definition 5.1 should be compared with the definition of a module (resp. balanced module) over an $A(m)$ -algebra (resp. balanced $A(m)$ -algebra) as it was given in [8, 1.10].

We next consider maps from $L(m)$ -algebras to differential graded Lie algebras.

Definition 5.2. *Let $L = (L, l_i)$ be an $L(m)$ -algebra and $A = (A, \partial_A, [-, -])$ a differential graded Lie algebra. A weak $L(m)$ -map from L to A is a collection $\{f_n \mid 1 \leq n \leq m - 1, n < \infty\}$ of skew symmetric linear maps $f_n : \otimes^n L \rightarrow A$ of degree $n - 1$ such that*

$$(8) \quad \partial_A f_n(\xi_1, \dots, \xi_n) + \sum_{j+k=n+1} \sum_{\sigma} \chi(\sigma) (-1)^{k(j-1)+1} f_j(l_k(\xi_{\sigma(1)}, \dots, \xi_{\sigma(k)}), \xi_{\sigma(k+1)}, \dots, \xi_{\sigma(n)}) \\ + \sum_{s+t=n} \sum_{\tau} \chi(\tau) (-1)^{s-1} \cdot (-1)^{(t-1)(\sum_{p=1}^s |\xi_{\tau(p)}|)} \cdot [f_s(\xi_{\tau(1)}, \dots, \xi_{\tau(s)}), f_t(\xi_{\tau(s+1)}, \dots, \xi_{\tau(n)})] = 0$$

where σ runs through all $(k, n - k)$ unshuffles and τ runs through all $(s, n - s)$ unshuffles such that $\tau(1) < \tau(s + 1)$, and $[-, -]$ denotes the graded bracket on A , $\xi_1, \dots, \xi_n \in L$.

Remark 5.3. Let $L = (L, l_i)$ and $L' = (L', l'_i)$ be two $L(\infty)$ -algebras. Let $({}^c\wedge W, \delta)$ and $({}^c\wedge W', \delta')$ be the corresponding differential graded coalgebras as in Theorem 2.3. We may say that a *weak map* from L to L' is a differential graded coalgebra homomorphism $\psi : ({}^c\wedge W, \delta) \rightarrow ({}^c\wedge W', \delta')$. We may also say that such a weak map is a (strict) *map* from L to L' if $\psi({}^c\wedge^n W) \subset {}^c\wedge^n W'$ for each $n \geq 1$. It is almost obvious to see that this definition is equivalent to the definition of a map as it was given in Section 2. Definition 5.2 is then equivalent in the special case $L' = A$, $l'_1 = \partial_A$, $l'_2 = [-, -]$, and $l'_k = 0$ for $k \geq 3$, to the definition of a weak map above. The case of a general $m < \infty$ can be discussed in a similar way.

In the special case of L having a strict differential graded Lie structure, our definition agrees with the definition of an $(m - 1)$ -homotopically multiplicative map studied by Retakh in [10].

The following theorem shows that we have the usual relationship between homomorphisms and module structures. Let $\text{End}(M)$ denote the graded associative algebra of linear maps from M to M with product given by composition and differential induced by the differential k_1 on M . Let us denote by $\text{End}(M)_L$ the differential graded Lie algebra associated to the differential graded associative algebra $\text{End}(M)$.

Theorem 5.4. *Suppose that $L = (L, l_i)$ is an $L(m)$ -algebra and that $M = (M, k_1)$ is a differential graded vector space. Then there exists a natural one-to-one correspondence between L -module structures on M and weak $L(m)$ -maps $L \rightarrow \text{End}(M)_L$.*

Proof. For each module structure map $k_n : \otimes^{n-1} L \otimes M \rightarrow M$ we define $f_{n-1} : \otimes^{n-1} L \rightarrow \text{End}(M)$ by

$$f_{n-1}(\xi_1, \dots, \xi_{n-1})(m) := (-1)^{n+1} \cdot k_n(\xi_1, \dots, \xi_{n-1}, m).$$

Let us consider the defining equation (7) multiplied by $(-1)^{n+1}$:

$$\sum_{i+j=n+1} \sum_{\sigma} \chi(\sigma) (-1)^{j(i+1)} \cdot k_j(k_i(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i)}), \xi_{\sigma(i+1)}, \dots, \xi_{\sigma(n)}) = 0.$$

We may obviously split the summation into four parts, the first one with $\sigma(n) = n$ and $j > 1$, the second one with $\sigma(i) = n$ and $i, j > 1$, the third one with $\sigma(i) = n$, $i = 1$ and $j > 1$, and the fourth one with $j = 1$. We obtain

$$\begin{aligned} 0 &= \sum_{\substack{i+j=n+1 \\ j>1}} \sum_{\sigma(n)=n} \chi(\sigma) (-1)^{j(i+1)} \cdot k_j(l_i(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i)}), \xi_{\sigma(i+1)}, \dots, \xi_n) \\ &+ \sum_{\substack{i+j=n+1 \\ j>1, i>1}} \sum_{\sigma(n) \neq n} \chi(\sigma) (-1)^{j(i+1)} \cdot k_j(k_i(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i)}), \xi_{\sigma(i+1)}, \dots, \xi_{\sigma(n)}) \\ &- (-1)^n (-1)^{|\xi_n|(\sum_{p=1}^{n-1} |\xi_{\sigma(p)}|)} \cdot k_n(k_1(\xi_n), \xi_1, \dots, \xi_{n-1}) + (-1)^{n+1} \cdot k_1(k_n(\xi_1, \dots, \xi_n)). \end{aligned}$$

The first term of the right-hand side may then be written as

$$\sum_{i+j=n+1} \sum_{\sigma} \chi(\sigma) (-1)^{ij+1} \cdot f_{j-1}(l_i(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i)}), \xi_{\sigma(i+1)}, \dots, \xi_{\sigma(n-1)})(\xi_n)$$

which is easily seen to correspond to the first part of (8) (after the substitution $n \mapsto n + 1$, $j \mapsto j + 1$ and $i \mapsto k$).

The second term of the relation requires a more subtle examination. For a fixed $(i, n - i)$ unshuffle σ with $\sigma(i) = n$, we have the corresponding $(n - i + 1, i - 1)$ unshuffle σ' with $\sigma'(n - i + 1) = n$ given by $\sigma' : (1, \dots, n) \mapsto (\sigma(i + 1), \dots, \sigma(n), \sigma(i), \sigma(1), \dots, \sigma(i - 1))$. We then pair the terms that are indexed by these two unshuffles and reindex the sum with just one of the unshuffles, say σ , where σ is chosen so that $\sigma(1) < \sigma(i + 1)$, to obtain

$$\begin{aligned} & \chi(\sigma)(-1)^{j(i+1)} \cdot k_j(k_i(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i)}), \xi_{\sigma(i+1)}, \dots, \xi_{\sigma(n)}) \\ & + \chi(\sigma)(-1)^{i(j+1)} \beta \cdot k_i(k_j(\xi_{\sigma(i+1)}, \dots, \xi_{\sigma(n)}, \xi_{\sigma(i)}), \xi_{\sigma(1)}, \dots, \xi_{\sigma(i-1)}) \end{aligned}$$

where

$$\beta = (-1)^{ij-1} \cdot (-1)^{|\xi_{\sigma(i)}| \cdot (\sum_{q=i+1}^n |\xi_{\sigma(q)}|) + (\sum_{p=1}^{i-1} |\xi_{\sigma(p)}|) (\sum_{q=i}^n |\xi_{\sigma(q)}|)}$$

defined by $\chi(\sigma') = \beta \cdot \chi(\sigma)$ is the sign adjustment that allows us to relate the two unshuffles to the same permutation σ .

We rewrite this sum as

$$\begin{aligned} & \chi(\sigma)(-1)^{j(i+1)} \cdot \alpha_1 \cdot k_j(\xi_{\sigma(i+1)}, \dots, \xi_{\sigma(n)}, k_i(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i)})) \\ & + \chi(\sigma)(-1)^{i(j+1)} \cdot \beta \alpha_2 \cdot k_i(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i-1)}, k_j(\xi_{\sigma(i+1)}, \dots, \xi_{\sigma(n)}, \xi_{\sigma(i)})) \end{aligned}$$

where

$$\alpha_1 = (-1)^{j-1} \cdot (-1)^{(i + \sum_{p=1}^i |\xi_{\sigma(p)}|) \cdot (\sum_{q=i+1}^n |\xi_{\sigma(q)}|)}$$

is defined by $\chi(k_i(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i)}), \xi_{\sigma(i+1)}, \dots, \xi_{\sigma(n)}) = \alpha_1 \cdot \chi(\xi_{\sigma(i+1)}, \dots, \xi_{\sigma(n)}, k_i(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i)}))$ and

$$\alpha_2 = (-1)^{i-1} \cdot (-1)^{(j + \sum_{q=i}^n |\xi_{\sigma(q)}|) \cdot (\sum_{p=1}^{i-1} |\xi_{\sigma(p)}|)}$$

is defined by

$$\begin{aligned} & \chi(k_j(\xi_{\sigma(i+1)}, \dots, \xi_{\sigma(n)}, \xi_{\sigma(i)}), \xi_{\sigma(1)}, \dots, \xi_{\sigma(i-1)}) = \\ & = \alpha_2 \cdot \chi(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i-1)}, k_j(\xi_{\sigma(i+1)}, \dots, \xi_{\sigma(n)}, \xi_{\sigma(i)})). \end{aligned}$$

When the correspondence $k_n \leftrightarrow (-1)^{n+1} \cdot f_{n-1}$ is made explicit in the above, we arrive at

$$\begin{aligned} & \chi(\sigma) \cdot \varphi \cdot \left\{ (f_{i-1}(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i-1)}) \circ f_{j-1}(\xi_{\sigma(i+1)}, \dots, \xi_{\sigma(n)}) \right. \\ & \quad \left. - \gamma f_{j-1}(\xi_{\sigma(i+1)}, \dots, \xi_{\sigma(n)}) \circ f_{i-1}(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i-1)}) \right\} (\xi_{\sigma(i)}) \end{aligned}$$

where $\varphi := (-1)^{i+j} \cdot (-1)^{|\xi_{\sigma(i)}| (\sum_{q=i+1}^n |\xi_{\sigma(q)}|)} \cdot (-1)^{j (\sum_{p=1}^{i-1} |\xi_{\sigma(p)}|)}$ and

$$\gamma = (-1)^{|f_{i-1}(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i-1)})| \cdot |f_{j-1}(\xi_{\sigma(i+1)}, \dots, \xi_{\sigma(n)})|} = (-1)^{(i + \sum_{p=1}^{i-1} |\xi_{\sigma(p)}|) \cdot (j + \sum_{q=i+1}^n |\xi_{\sigma(q)}|)}$$

is the sign required for the commutator. Let us define an $(i-1, n-i-1)$ -unshuffle τ by

$$\tau(k) := \begin{cases} \sigma(k), & \text{for } 1 \leq k \leq i-1, \text{ and} \\ \sigma(k+1), & \text{for } i \leq k \leq n-1. \end{cases}$$

Then $\chi(\sigma) = (-1)^j \cdot (-1)^{|\xi_{\sigma(i)}| \cdot (\sum_{q=i+1}^n |\xi_{\sigma(q)}|)} \cdot \chi(\tau)$ and the substitution $\sigma \mapsto \tau$ enables us to write the above expression as

$$\chi(\tau) \cdot (-1)^{j \cdot (\sum_{p=1}^{i-1} |\xi_{\tau(p)}|)} \cdot (-1)^i \cdot [f_{i-1}(\xi_{\tau(1)}, \dots, \xi_{\tau(i-1)}), f_{j-1}(\xi_{\tau(i)}, \dots, \xi_{\tau(n-1)})](\xi_n)$$

which corresponds, after the substitution $i \mapsto s+1$, $j \mapsto t+1$ and $n \mapsto n+1$, to the third term of (8).

The remaining two terms can be written as

$$k_1 \circ f_{n-1}(\xi_1, \dots, \xi_{n-1})(\xi_n) - (-1)^{(n + \sum_{k=1}^{n-1} |\xi_k|)} \cdot f_{n-1}(\xi_1, \dots, \xi_{n-1})(k_1(\xi_n))$$

which is the differential in $\text{End}(M)_L$ applied to $f_{n-1}(\xi_1, \dots, \xi_{n-1})$, i.e. the first term of (8) (after the substitution $n \mapsto n+1$). ■

We note that the pairing of the unshuffles in the above proof leads to the same index set called "regular sequences" in [10].

We believe that an analog of Theorem 5.4 holds also for modules (resp. balanced modules) over an $A(m)$ -algebra (resp. balanced $A(m)$ -algebra) [8, 1.10].

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