

INTRODUCTION TO SH LIE ALGEBRAS FOR PHYSICISTS

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Much of point particle physics can be described in terms of Lie algebras and their representations. Closed string field theory, on the other hand, leads to a generalization of Lie algebra which arose naturally within mathematics in the study of deformations of algebraic structures [SS]. It also appeared in work on higher spin particles [BBvD]. Representation theoretic analogs arose in the mathematical analysis of the Batalin-Fradkin-Vilkovisky approach to constrained Hamiltonians [S6].

The sh Lie algebra of closed string field theory [SZ], [KKS], [K], [Wies], [WZ], [Z] is defined on the full Fock complex of the theory, with the BRST differential Q . Following Zwiebach [Z], we stipulate that the string fields B_1, B_2, \dots are elements of \mathcal{H} , the Hilbert space of the combined conformal field theory of matter and ghosts. The (n -fold) string product (bracket) for genus 0 is denoted by

$$[B_1, B_2, \dots, B_n]_0.$$

It has n entries, that is, n states in \mathcal{H} . Since we will deal only with genus 0, we will omit the subscript 0 henceforth. The basic equation relating these brackets and the BRST operator is:

(1)

$$0 = Q[B_1, \dots, B_n] + \sum_{i=1}^n \pm [B_1, \dots, QB_i, \dots, B_n] + \sum \sigma(,) [B_{i_1}, \dots, B_{i_j}, [B_{i_{j+1}}, \dots, B_n]],$$

where the second sum is over all unshuffles (see below) and $\sigma(,)$ denotes an appropriate sign.

In their work on higher-spin particles, Berends, Burgers and van Dam consider infinitesimal gauge transformations of the form

$$\delta_\xi \phi = \partial \xi + \sum_{n=2}^{\infty} g^n T_n(\phi, \xi)$$

where T_n is n -linear in ϕ and linear in ξ and g is a “coupling constant”. Notice the unusual ϕ -dependence of the transformation. They consider the case in which the commutator of two such transformations is again of this form, which they use to define

$$[\xi_1, \xi_2] = \sum_{n=0}^{\infty} g^{n+1} C_n(\phi, \xi_1, \xi_2)$$

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where C_n is n -linear in ϕ and linear in each ξ_i . For this bracket to satisfy the Jacobi identity implies a whole sequence of conditions for each order g^n . In particular, if the ξ_i and ϕ_j are fields of the same sort and we take $C_n = T_{n+1}$, then to order g^2 they find:

$$\partial T_2(\phi_1, \phi_2, \phi_3) = \sum_{i=1}^3 \pm T_2(\cdots, \partial \phi_i, \cdots) + \sum \pm T_1(T_1(\phi_{i_1}, \phi_{i_2}), \phi_{i_3})$$

while to higher order, Burgers [B] finds the full analogs of (1).

To see the above formulas as a generalization of those for a differential graded Lie algebra is the major goal of this paper, hopefully describing the mathematical essentials in terms accessible to *physicists*.

The concept of Lie algebra can be expressed in several different ways. The most familiar are: in terms of generators and relations and in terms of a bilinear ‘‘bracket’’ satisfying the Jacobi identity. In ‘‘physical’’ notation, let X_a be a basis for V . The bracket $[\ , \]$ can be specified by structure constants C_{ab}^c via the formula:

$$[X_a, X_b] = C_{ab}^c X_c.$$

The structure constants are skew-symmetric in the lower indices a, b .

A much more subtle description appears in the homological study of Lie algebras, but it is this description which is the most useful in homological perturbation theory and in mathematical physics in the guise of BRST operators for open algebras and in the algebra for closed string field theory. This description is implicit in the somewhat more familiar dual formulation of the Chevalley-Eilenberg cochain complex for Lie algebra cohomology [CE]: An **n-cochain** is a skew-symmetric n -linear function $\omega : V \times \cdots \times V \rightarrow \mathbb{R}$ and the coboundary $d\omega$ is defined by

$$d\omega(v_1, \dots, v_{n+1}) = \sum_{i < j} (-1)^{i+j} \omega([v_i, v_j], v_1, \dots, v_1, \dots, \hat{v}_i \dots \hat{v}_j \dots, v_{n+1})$$

where the ‘‘hatted’’ variables are to be omitted. With respect to a dual basis ω^a of the dual vector space V^* , we can write d as $-1/2 \omega^a \omega^b C_{ab}^c \partial_{X_c}$.

We can deal directly with the vectors rather than with the multi-linear ‘forms’ at the expense of introducing a new point of view and consideration of skew-symmetric tensors (*multi-vectors*).

A Lie algebra is equivalent to the following data:

A vector space V (assumed finite dimensional for simplicity of exposition).

The skew (or alternating) tensor products of V , denoted

$$\bigwedge^n V = \left\{ \bigwedge^n V \right\}.$$

A linear map

$$d : \bigwedge^n V \rightarrow \bigwedge^{n-1} V$$

which lowers n by one and is a *co-derivation* such that $d^2 = 0$.

(That d is a **co-derivation** means just that

$$d(v_1 \wedge \cdots \wedge v_n) = \sum (-1)^{i+j} d(v_i \wedge v_j) \wedge v_1 \wedge \cdots \wedge \hat{v}_i \cdots \wedge \hat{v}_j \cdots \wedge v_n.$$

For example,

$$d(X_a \wedge X_b \wedge X_c) = -C_{ab}^e X_e \wedge X_c + C_{ac}^e X_e \wedge X_b - C_{bc}^e X_e \wedge X_a.)$$

It may not be immediately obvious, but d restricted to \bigwedge^2 is to be interpreted as a bracket: $d(v_1 \wedge v_2) = [v_1, v_2]$ and $d^2 = 0$ is equivalent to the Jacobi identity.

From the point of view of the skew tensor powers of V , an **sh Lie algebra (strongly homotopy Lie algebra)** is similarly equivalent to a straightforward generalization in which d is replaced by a co-derivation

$$D = d_1 + d_2 + d_3 + \dots$$

where d_i lowers n by $i - 1$, in particular, $d_n(v_1 \wedge \dots \wedge v_n) \in V$. We say that D is a coderivation to summarize the several conditions:

$$d_i(v_1 \wedge \dots \wedge v_n) = \sum \pm d_i(v_{i_1} \wedge \dots \wedge v_{i_j}) \wedge v_{i_{j+1}} \wedge \dots \wedge v_{i_n}$$

where the sum is over all *unshuffles* of $\{1, \dots, n\}$, that is, all permutations that keep i_1, \dots, i_j and i_{j+1}, \dots, i_n in the same relative order. A *shuffle* of two ordered sets (decks of cards) is a permutation of the ordered union which preserves the order of each of the given subsets; an *unshuffle* reverses the process, cf. Maxwell's demon. (The serious issue of signs will be addressed below.)

Notice that the old d corresponds to d_2 since $d(v_1 \wedge v_2) = [v_1, v_2] \in V$. On the other hand, for the new D , the component d_2 no longer is of square zero by itself and hence corresponds to a bracket which does NOT necessarily satisfy the Jacobi identity. Let us look in detail at what can happen instead:

Expand $D^2 = 0$ in its homogeneous components and set them separately equal to zero. We have then:

$$0) \quad d_1^2 = 0$$

so (V, d_1) is a complex or differential (graded) module. (Usually V is itself a graded module and D is a *graded* co-derivation, which implies appropriate signs in applying D to $x_1 \wedge \dots \wedge x_n$. Also d_1 typically raises (or lowers) that 'internal' degree by 1.) *Perhaps the most important examples in physics are the exterior derivative on differential forms and the BRST operator on suitable fields, whether physical or ghostly.*

$$1) \quad d_1 d_2 + d_2 d_1 = 0$$

so, with appropriate sign conventions, d_2 gives a bracket $[v_1, v_2] \in V$ for which d_1 is a derivation.

$$2) \quad d_1 d_3 + d_2 d_2 + d_3 d_1 = 0$$

or equivalently

$$d_2 d_2 = -(d_1 d_3 + d_3 d_1).$$

If we further adopt the notation:

$$d(x_1 \wedge x_2 \wedge x_3) = [x_1, x_2] \wedge x_3 + \dots$$

then we have

$$[[v_1, v_2], v_3] \pm [[v_1, v_3], v_2] \pm [[v_2, v_3], v_1] = -d_1[v_1, v_2, v_3] \pm [d_1 v_1, v_2, v_3] \pm [v_1, d_1 v_2, v_3] \pm [v_1, v_2, d_1 v_3].$$

From this, with use of the skew-symmetry of $[\ , \]$, we see that now the Jacobi identity holds modulo the right hand side. *In physical language, the Jacobi identity holds modulo a BRST exact term.* In the language of homological algebra, d_3 is a chain homotopy, so we say that (V, d_2) satisfies the Jacobi identity *up to homotopy* or (V, d_1, d_2, d_3) is a **homotopy Lie algebra**. The adverb “**strongly**” is added to refer to the other d_i .

In physical notation, restricting d_3 to $\bigwedge^3 V$, we can write:

$$d_3(X_a \wedge X_b \wedge X_c) = C_{abc}^e X_e$$

where C_{abc}^e is skew-symmetric in the lower indices. Similarly, we can write

$$d_1 X_a = C_a^b X_b.$$

Just as the Jacobi identity can be written as a quadratic equation in the C_{ab}^c , so equation 2) can be written as a quadratic equation in the C_a^b , C_{ab}^c and C_{abc}^e .

If we adopt the notation that $d_1 = Q$ and in general:

$$d_n(x_1 \wedge \cdots \wedge x_n) = [x_1, x_2, \dots, x_n],$$

then the appropriate homogeneous piece of $D^2 = 0$ is (up to sign conventions and up to some constants related to conventions on the definition of $\bigwedge V$) precisely the equation (1) occurring in the ‘non-polynomial’ version of the genus zero closed string field algebra: The sum is not only over j but also over all $(j, n-j)$ unshuffles. Making the correspondence precise, including the appropriate signs, requires some care, as in the next section.

In the higher spin particle algebra of **[BBvD]**, variations δ_ϵ do not respect a strict bracket $[\epsilon_1, \epsilon_2]$ but rather an sh Lie structure on the space of ϵ ’s. In the Batalin-Fradkin-Vilkovisky operator for constraints forming an ‘open’ algebra with structure functions, one sees a similar structure **[S6]**.

This paper is organized as follows: After establishing our notation and conventions (especially for signs), we give the formal definition of sh Lie structure and verify the equivalence with a formulation in terms of a ‘nilpotent’ operator on $\bigwedge sV$. Comparison with the physics literature calls attention to some further subtleties of signs. Then we establish the sh analog of the familiar fact that commutators in an associative algebra form a Lie algebra. We next point out the relevance of these structures to $N + 1$ -point functions in physics. We remark on the distinction between these structures on the cohomology level and at the underlying form level and conclude with the basic theorem of Homological Perturbation Theory relating higher order bracket operations on cohomology to strict Lie algebra structures on forms.

Recognition that the mathematical structure of sh Lie algebras was appearing in physics first occurred in discussions with Burgers and van Dam and then with Zwiebach at the last GUT Workshop. He called our attention to the relevant preprints **[SZ]** and **[KKS]** and has been a continued source of inspiration for the present paper, cf. especially **[Z]**. Our attempts to make this exposition accessible to physicists have benefitted also from comments of Henrik Aratyn, José Figueroa-O’Farrill and Takashi Kimura. We are indebted to Elizabeth Jones Dempster for some of the hard technical details with regard to signs, especially in comparing sh Lie structures to sh associative ones.

Notation and Conventions.

In dealing with maps of differential graded vector spaces, it is crucial to keep careful track of signs. First, there are the appropriate signs for any graded (or *super*) context. The basic convention is that whenever two symbols, of degree m and n respectively, are interchanged, a sign of $(-1)^{mn}$ is introduced. In particular, if σ is a permutation that is acting on a string of such symbols, $e(\sigma)$ will denote the sign that results from iteration of the basic convention, i.e. $e(\sigma) = (-1)^k$ where k is the number of interchanges of odd symbols. We also let $(-1)^\sigma$ denote the usual sign of the permutation σ . It is important to note that $e(\sigma)$ does not involve $(-1)^\sigma$ as a factor. However, if all the elements in question have degree 1, then $e(\sigma) = (-1)^\sigma$.

We always regard the symmetric group \mathcal{S}_n as acting on $V^{\otimes n}$ by

$$\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

A map of graded vector spaces $f : V^{\otimes n} \rightarrow W$ is called

symmetric if $f(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}) = e(\sigma)f(v_1 \otimes \cdots \otimes v_n)$

and

skew symmetric if $f(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}) = (-1)^\sigma e(\sigma)f(v_1, \dots, v_n)$

for all $\sigma \in \mathcal{S}_n$. In terms of a basis for V , we can write f in terms of its coefficients $f^{a_1 \dots a_n}$, then symmetric or skew symmetric has the usual interpretation with respect to permutations of $a_1 \dots a_n$.

Now let $V = \{V^i\}$ be a graded vector space over a field k . Let $T^*(V)$ denote the tensor vector space generated by V , i.e. $\{V^{\otimes n}\}$. We do NOT consider $T^*(V)$ as the tensor algebra, but rather as a **coalgebra** with the standard coalgebra structure given by the diagonal

$$\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_{j=0}^n (v_1 \otimes \cdots \otimes v_j) \otimes (v_{j+1} \otimes \cdots \otimes v_n).$$

(Here $V^{\otimes 0}$ is to be identified with k and the terms with $j = 0$ or $j = n$ are of the form $1 \otimes (\dots)$ and $(\dots) \otimes 1$ respectively.) The use of coalgebras is an efficient device for some of our expressions, but it is sufficient to follow the argument in terms of ordinary tensors.

In particular, we will make use of $\bigwedge V$ which consists of the skew-symmetric tensors (in the graded sense), that is the subspace (in fact, sub-coalgebra) of $T^*(V)$ which is fixed under the above action of the symmetric group on $V^{\otimes n}$. Although we will not need the terminology, $\bigwedge V$ is known as the **free cocommutative coalgebra** generated by V with reduced diagonal given by

$$\bar{\Delta}(u_1 \wedge \cdots \wedge u_n) = \sum_{j=1}^{n-1} \sum_{\sigma} e(\sigma)(u_{\sigma(1)} \wedge \cdots \wedge u_{\sigma(j)}) \otimes (u_{\sigma(j+1)} \wedge \cdots \wedge u_{\sigma(n)})$$

where σ runs through all $(j, n-j)$ unshuffles

sh Lie Structures.

Let V be a differential graded vector space with differential denoted by $l_1 : V_i \rightarrow V_{i-1}$. We first recall the graded version of an ordinary Lie algebra.

Definition. A **graded Lie algebra** is a graded vector space $V = \{V_p\}$ together with a graded skew commutative bracket $[\cdot, \cdot] : V \otimes V \rightarrow V$ such that $V_p \otimes V_q \rightarrow V_{p+q}$ satisfying the graded Jacobi identity:

$$[u, [v, w]] = [[u, v], w] + (-1)^{pq}[v, [u, w]].$$

An alternative “with the grading reduced by one” (meaning $V_p \otimes V_q \rightarrow V_{p+q-1}$) occurs in the Hochschild cohomology of an associative algebra as part of the structure of a Gerstenhaber algebra **[G]**, a structure which is also making an appearance in physics.

Examples have existed for a long time. Perhaps the earliest is the Schouten bracket of multivectors although it was not identified as such until long after its introduction. Similarly the Whitehead product of homotopy groups **[Wh]** (which has the reduced grading) and the corresponding Samelson product **[Sa]** were also not so identified initially.

We will also consider the differential graded vector space sV which is defined by $(sV)_i = V_{i-1}$ with $\hat{l}_1(sv) = -sl_1(v)$. The use of such a suspension operator s is implicit in the Chevalley-Eilenberg complex, but for the more complicated sh Lie structures, it is best to make it explicit. For similar reasons, especially subtlety of signs, and also for comparison with the corresponding sh associative algebras, we have chosen to express this section in terms of maps l_n and \hat{l}_n rather than in terms of the d_n of the introduction.

Definition. An **sh Lie structure** on V is a collection of skew symmetric linear maps $l_n : \bigotimes^n V \rightarrow V$ of degree $2 - n$ (for cochain complexes, and $n - 2$ for chain complexes) such that

$$(2) \quad \sum_{i+j=n+1} \sum_{\sigma} e(\sigma)(-1)^{\sigma} \alpha(i, j) l_i(l_j(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(j)}) \otimes v_{\sigma(j+1)} \otimes \cdots \otimes v_{\sigma(n)}) = 0$$

where $\alpha(i, j) = -1$ only if i is odd and j is even and 1 otherwise, while σ runs through all $(j, n - j)$ unshuffles (cf. **[J]**).

It is clear that we may use the skew symmetry of the l_n 's to induce linear maps $l_n : \bigwedge^n V \rightarrow V$ that satisfy the same equations in the definition. We note that the map l_2 may be viewed as a usual (graded) Lie bracket and that when $n = 3$ and $l_3 = 0$, the definition yields the usual (graded) Jacobi identity. In general, l_3 is a homotopy between the Jacobi expression and 0 while the other l_n 's are higher homotopies.

It is worth calling attention to the matter of degrees, both for the forms (e.g. *ghost degree in the BRST context*) and the degree of the N -fold operations. The original mathematical formulation as above) of sh Lie algebras (and their sh associative algebra predecessors) assumed that the (2-fold) product or bracket had degree zero, i.e. the degree of the bracket was the sum of the degrees of the factors and that the degree of $d = d_1$ was 1 (for cohomology) or -1 (for homology). The defining equality (1) then determines the degree of the other N -fold brackets. If

the bracket should itself have a degree (for example, -1 for the Whitehead product or the Gerstenhaber bracket or 1 in some physical examples), then the degrees of the other N -fold operations would be adjusted accordingly, as would the signs in (2). In particular, as Zwiebach is careful to point out in section 4.1 of [Z], in closed string field theory, the degree is given by the statistics, so the degree of the 2-fold bracket is 1. As he explains in detail, a simple shift in counting degrees and appropriate changes in signs establishes the equivalence of our two sets of conventions. This is precisely the shift that relates the Whitehead product to the Samelson product [Wh], [Sa].

We now use the graded vector space sV ; recall $(sV)_i = V_{i-1}$. Define linear maps $\hat{l}_n : \bigwedge^n sV \longrightarrow sV$ by

$$(3) \quad \hat{l}_n(sv_1 \wedge \cdots \wedge sv_n) = \begin{cases} (-1)^{\sum_{i=1}^{n/2} |v_{2i-1}|} sl_n(v_1 \wedge \cdots \wedge v_n), & n \text{ even} \\ -(-1)^{\sum_{i=1}^{(n-1)/2} |v_{2i}|} sl_n(v_1 \wedge \cdots \wedge v_n), & n \text{ odd} \end{cases}$$

where $|v_k| = \text{degree of } v_k$.

We may now further extend $\hat{l}_n : \bigwedge^k sV \longrightarrow \bigwedge^{k-n+1} sV$ as a coderivation with respect to the usual coproduct on $\bigwedge^* sV$. Explicitly, we have

$$(4) \quad \hat{l}_n(sv_1 \wedge \cdots \wedge sv_k) = \sum_{\sigma} e(\sigma) \hat{l}_n(sv_{\sigma(1)} \wedge \cdots \wedge sv_{\sigma(n)}) \wedge sv_{\sigma(n+1)} \wedge \cdots \wedge sv_{\sigma(k)}$$

where σ runs through all $(n, k-n)$ unshuffles. Define the degree of an element in $\bigwedge^* sV$ by $|sv_1 \wedge \cdots \wedge sv_k| = k + \sum_{i=1}^k |v_i|$. Note that with this grading each \hat{l}_n has degree -1 . The ordering convention in (4) with the \hat{l}_n terms listed first provides a convenient way of keeping track of the signs and the unshuffles. We observe however that if an element of $\bigwedge^k sV$ has the form

$$sv_{\sigma(1)} \wedge \cdots \wedge sv_{\sigma(i)} \wedge sv_{\sigma(i+1)} \wedge \cdots \wedge sv_{\sigma(i+n)} \wedge \cdots \wedge sv_{\sigma(k)}$$

where the sequence $\sigma(i+1), \dots, \sigma(i+n)$ respects the original ordering of the sv_i 's, we then have among the summands in (4) an equivalent term of the form

$$(-1)^{(\sum_{p=1}^i |sv_{\sigma(p)}|)(|\hat{l}_n|)} e(\sigma) sv_{\sigma(1)} \wedge \cdots \wedge sv_{\sigma(i)} \wedge \hat{l}_n(sv_{\sigma(i+1)} \wedge \cdots \wedge sv_{\sigma(i+n)}) \wedge \cdots \wedge sv_{\sigma(k)}$$

by use of the basic sign conventions.

In order to calculate compositions of the l_n 's we require the following lemma.

Lemma. $\hat{l}_i(\hat{l}_j(sv_1 \wedge \cdots \wedge sv_j) \wedge sv_{j+1} \wedge \cdots \wedge sv_{j+i-1}) =$

$$\begin{aligned} & (-1)^{\sum_{p=1}^{i+j-2/2} |v_{2p}|} sl_i(l_j(v_1 \wedge \cdots \wedge v_j) \wedge v_{j+1} \wedge \cdots \wedge v_{i+j-1}) \text{ if } i+j \text{ is even} \\ & (-1)^{\sum_{p=1}^{i+j-1/2} |v_{2p-1}|} sl_i(l_j(v_1 \wedge \cdots \wedge v_j) \wedge v_{j+1} \wedge \cdots \wedge v_{i+j-1}) \text{ if } i \text{ is even and } j \text{ is odd} \\ & (-1)^{1+\sum_{p=1}^{i+j-1/2} |v_{2p-1}|} sl_i(l_j(v_1 \wedge \cdots \wedge v_j) \wedge v_{j+1} \wedge \cdots \wedge v_{i+j-1}) \text{ if } j \text{ is even and } i \text{ is odd.} \end{aligned}$$

Proof. We first assume that both $i+j$ and j are even. Then by using (3), we have

$$\begin{aligned} & \hat{l}_i(\hat{l}_j(sv_1 \wedge \cdots \wedge sv_j) \wedge sv_{j+1} \wedge \cdots \wedge sv_{i+j-1}) = \\ & (-1)^{\sum_{p=1}^{j/2} |v_{2p-1}|} (-1)^{|l_j(v_1 \wedge \cdots \wedge v_j)| + \sum_{p=1}^{i-2/2} |v_{j+2p}|} sl_i(l_j(v_1 \wedge \cdots \wedge v_j) \wedge v_{j+1} \wedge \cdots \wedge v_{i+j-1}). \end{aligned}$$

Note that the sign in the above expression is equal to

$$(-1)^{\sum_{p=1}^{j/2} |v_{2p-1}|} (-1)^{j-2+\sum_{p=1}^j |v_p| + \sum_{p=1}^{i-2/2} |v_{j+2p}|} = (-1)^{\sum_{p=1}^{i+j-2} |v_{2p}|}$$

which is what is claimed. The other three cases follow from a similar calculation. \square

Theorem. Let $D : \Lambda^* sV \longrightarrow \Lambda^* sV$ be given by $D = \sum_i \hat{l}_i$. Then $D^2 = 0$.

Note: This gives a generalization of the Chevalley-Eilenberg complex for a Lie algebra. The rest of this section is devoted to a proof which amounts to checking that the signs cancel appropriately.

Proof. Evaluate $D^2 = \sum_{k=2}^{n+1} \sum_{i+j=k} \hat{l}_i \hat{l}_j$ on $sv_1 \wedge \cdots \wedge sv_n$. In fact we claim that for each $k \leq n+1$, we have $\sum_{i+j=k} \hat{l}_i \hat{l}_j (sv_1 \wedge \cdots \wedge sv_n) = 0$. We use (4) to write

$$\hat{l}_i \hat{l}_j (sv_1 \wedge \cdots \wedge sv_n) = \hat{l}_i \left(\sum_{\sigma} e(\sigma) \hat{l}_j (sv_{\sigma(1)} \wedge \cdots \wedge sv_{\sigma(j)}) \wedge sv_{\sigma(j+1)} \wedge \cdots \wedge sv_{\sigma(n)} \right).$$

Now use (4) again to evaluate \hat{l}_i . At this point we note that a typical summand of the resulting expression will have the form

$$(a) \ e(\sigma) \hat{l}_i (\hat{l}_j (sv_{\sigma(1)} \wedge \cdots \wedge sv_{\sigma(j)}) \wedge sv_{\sigma(j+1)} \wedge \cdots \wedge sv_{\sigma(j+i-1)}) \wedge sv_{\sigma(j+i)} \wedge \cdots \wedge sv_{\sigma(n)})$$

or

$$(b) \ e(\sigma) \hat{l}_i (sv_{\sigma(1)} \wedge \cdots \wedge sv_{\sigma(i)}) \wedge \hat{l}_j (sv_{\sigma(i+1)} \wedge \cdots \wedge sv_{\sigma(i+j)}) \wedge sv_{\sigma(i+j+1)} \wedge \cdots \wedge sv_{\sigma(n)})$$

Here σ refers to a composition of unshuffles and $e(\sigma)$ is the resulting product of their signs.

We begin with the type (a) terms. First collect all terms together that have identical last $n - j - i + 1$ entries and denote this sequence of terms by sv_Q . Then

$$\sum_{i+j=k} \sum_{\sigma} e(\sigma) \hat{l}_i \hat{l}_j (sv_{\sigma(1)} \wedge \cdots \wedge sv_{\sigma(j)}) \wedge sv_{\sigma(j+1)} \wedge \cdots \wedge sv_{\sigma(j+i-1)} \wedge sv_Q =$$

(5)

$$(-1)^{\sum_{p=1}^{i+j-2/2} |v_{2p}|_s} \left(\sum_{i+j=k} \sum_{\sigma} e(\sigma) (-1)^{\sigma} l_i (l_j (v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(j)}) \wedge v_{\sigma(j+1)} \wedge \cdots \wedge v_{\sigma(j+i-1)}) \wedge sv_Q \right)$$

if $i+j$ is even, and

(6)

$$(-1)^{\sum_{p=1}^{i+j-2/2} |v_{2p-1}|_s} \left(\sum_{i+j=k} \sum_{\sigma} e(\sigma) (-1)^{\sigma} (-1)^i l_i (l_j (v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(j)}) \wedge v_{\sigma(j+1)} \wedge \cdots \wedge v_{\sigma(j+i-1)}) \wedge sv_Q \right)$$

if $i+j$ is odd. Here σ is taken over all $(j, i-1)$ unshuffles in the first $j+i-1$ coordinates. Both (5) and (6) are equal to 0 by (2).

To verify this claim we use the fact that each unshuffle is the product of transpositions, and examine the effect of one transposition on the term

$$\hat{l}_i (\hat{l}_j (sv_1 \wedge \cdots \wedge sv_j) \wedge sv_{j+1} \wedge \cdots \wedge sv_{i+j-1})$$

while using the lemma to evaluate $\hat{l}_i \hat{l}_j$. In fact, the transposition that we need consider is $\sigma = (j, j+1)$. Consequently, we have

$$(-1)^{|sv_i|_s |sv_{i+1}|_s} \hat{l}_i (\hat{l}_j (sv_1 \wedge \cdots \wedge sv_{j-1}) \wedge sv_j \wedge sv_{j+1} \wedge \cdots \wedge sv_{i+j-1})$$

$$(-1)^{|v_j||v_{j+1}|+1+|v_j|+|v_{j+1}|}\hat{l}_i(\hat{l}_j(sv_1 \wedge \cdots \wedge sv_{j+1}) \wedge sv_j \wedge \cdots \wedge sv_{i+j-1}) =$$

$$e(\sigma)(-1)^\sigma(-1)^{|v_j||v_{j+1}|}(-1)^{\sum_{p=1}^{j+i-2/2} |v_{\sigma(2p)}|} sl_i(l_j(v_1 \wedge \cdots \wedge v_{\sigma(j)}) \wedge v_{\sigma(j+1)} \wedge \cdots \wedge v_{i+j-1})$$

by the lemma when $i + j$ is even. We note that since $\sigma(j) = j + 1$ and since j and $j + 1$ have opposite parity, the exponent

$$|v_j| + |v_{j+1}| + \sum_{p=1}^{i+j-2/2} |v_{\sigma(2p)}| = \sum_{p=1}^{i+j-2/2} |v_{2p}|.$$

The case in which $i + j$ is odd follows by a similar calculation.

We now turn our attention to the terms of type (b). These terms occur in pairs with opposite signs and thus cancel each other out. They will occur when the sequences $\sigma(1), \dots, \sigma(i)$ and $\sigma(i + 1), \dots, \sigma(i + j)$ are ordered with respect to the usual ordering on the integers. We then evaluate

$$\hat{l}_i \hat{l}_j(sv_{\sigma(1)} \wedge \cdots \wedge sv_{\sigma(i)} \wedge sv_{\sigma(i+1)} \wedge \cdots \wedge sv_{\sigma(i+j)} \wedge sv_Q) =$$

$$(-1)^{(\sum_{p=1}^i |sv_{\sigma(p)}|)|\hat{l}_j|} \hat{l}_i(sv_{\sigma(1)} \wedge \cdots \wedge sv_{\sigma(i)} \wedge \hat{l}_j(sv_{\sigma(i+1)} \wedge \cdots \wedge sv_{\sigma(i+j)}) \wedge sv_Q) =$$

$$(-1)^{\sum_{p=1}^i |sv_{\sigma(p)}|} \hat{l}_i(sv_{\sigma(1)} \wedge \cdots \wedge sv_{\sigma(i)}) \wedge \hat{l}_j(sv_{\sigma(i+1)} \wedge \cdots \wedge sv_{\sigma(i+j)}) \wedge sv_Q.$$

On the other hand,

$$\hat{l}_j \hat{l}_i(sv_{\sigma(1)} \wedge \cdots \wedge sv_{\sigma(i+j)} \wedge sv_Q) =$$

$$\hat{l}_j(\hat{l}_i(sv_{\sigma(1)} \wedge \cdots \wedge sv_{\sigma(i)}) \wedge sv_{\sigma(i+1)} \wedge \cdots \wedge sv_{\sigma(i+j)} \wedge sv_Q) =$$

$$(-1)^{(\sum_{p=1}^i |sv_{\sigma(p)}|)|\hat{l}_j|} (-1)^{|\hat{l}_i||\hat{l}_j|} \hat{l}_i(sv_{\sigma(1)} \wedge \cdots \wedge sv_{\sigma(i)}) \wedge \hat{l}_j(sv_{\sigma(i+1)} \wedge \cdots \wedge sv_{\sigma(i+j)}) \wedge sv_Q.$$

Since $|\hat{l}_k| = -1$, the signs are as claimed. Note that in the evaluation of \hat{l}_i and \hat{l}_j here, we were only interested in the summands listed. \square

Commutators in relation to the sh associative case.

Physicists often refer to Lie brackets as commutators because commutators of elements in an associative algebra define a bracket satisfying the axioms for a Lie algebra: $[x, y] = xy - yx$ or, in the graded case, $[x, y] = xy - (-1)^{|x||y|}yx$. There is a notion of strongly homotopy associative (sha) algebra, much older than that of sh Lie algebra (cf. [S3, 5]); it is natural to try to put into the homotopy context the canonical construction of a Lie algebra from an associative algebra by use of commutators.

Definition. An **strongly homotopy associative (sha) structure** on V is a collection of linear maps $m_n : \bigotimes^n V \rightarrow V$ of degree $2 - n$ (for cochain complexes, and $n - 2$ for chain complexes) such that

$$(7) \quad \sum_{i+j=n+1} \sum_k \beta(i, j, k) m_i(v_1 \otimes \cdots \otimes v_{k-1} \otimes m_j(v_k \otimes \cdots \otimes v_{k+j-1}) \otimes \cdots \otimes v_n) = 0$$

where $\beta(i, j, k)$ is the sign given by the parity of $(j + 1)k + j(n + \sum_{m=1}^{k-1} |x_m|)$.

Definition. Given a differential graded vector space V with an sha structure $\{m_n\}$, the **commutator sh Lie structure** is defined on V by

$$l_n(v_1 \otimes \cdots \otimes v_n) = \sum_{\sigma} (-1)^{\sigma} e(\sigma) m_n(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)})$$

where the sum is taken over all permutations $\sigma \in S_n$.

When $n = 2$, $l_2(v_1 \otimes v_2) = m_2(v_1 \otimes v_2) - (-1)^{|v_1||v_2|} m_2(v_2 \otimes v_1)$ is the graded commutator. The possible lack of associativity of m_2 will in general prevent the Jacobi expression

$$(8) \quad l_2(l_2(v_1 \otimes v_2) \otimes v_3) - (-1)^{|v_2||v_3|} l_2(l_2(v_1 \otimes v_3) \otimes v_2) + (-1)^{|v_1|(|v_2|+|v_3|)} l_2(l_2(v_2 \otimes v_3) \otimes v_1)$$

from being equal to zero. However, with $l_3(v_1 \otimes v_2 \otimes v_3) = \sum_{\sigma \in S_3} (-1)^{\sigma} e(\sigma) m_3(v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)})$, one may check that (8) is equal to $l_1 l_3 + l_3 l_1$ where l_1 is defined to be m_1 . These details, as well as those required for 4-tensors are made explicit by Elizabeth Jones Dempster in [J]. The general verification that the above defined ‘commutator’ l_n ’s satisfy the defining equation for an sh Lie structure will appear elsewhere.

N+1-point functions.

As a generalization of Lie algebras, sh Lie algebras appear in closed string field theory as symmetries or gauge transformations. The corresponding Lagrangians consist of (sums of) $N + 1$ -point functions for all N (cf. [SZ], [KKS], [K], [Wies], [Z], [WZ]). They can be regarded as being formed from the N -fold brackets $[x_1, x_2, \dots, x_N]$, by evaluation with a dual field via an inner product or the $N + 1$ -point functions can be described directly. The latter is more appropriate for Kontsevich’s new invariants [Kon], but we will leave it to him to present those details. We adopt Dirac’s bra-ket notation and so write $|x\rangle$ instead of just x . We then write $\langle x|$ for elements of the dual space. In terms of a basis x_α or rather $|x_\alpha\rangle$, we have a dual basis $\langle x_\alpha|$ with

$$\langle x_\alpha | x_\beta \rangle = \delta_{\alpha\beta}.$$

In terms of the N -fold bracket, we then define

$$\{y_0 y_1 \cdots y_N\} = \langle y_0 | [y_1, y_2, \dots, y_N] \rangle.$$

Zwiebach streamlines the machinery in [KKS], giving the classical action in closed string field theory, the gauge transformations and showing the invariance of the action. The classical string action is simply given by

$$S(\Psi) = \frac{1}{2} \langle \Psi, Q\Psi \rangle + \sum_{n=3}^{\infty} \frac{\kappa^{n-2}}{n!} \{\Psi \dots \Psi\}.$$

The expression $\{\Psi \dots \Psi\}$ contains n -terms and will be abbreviated $\{\Psi^n\}$, and similarly for $[\Psi^n, \Lambda]$ below.

The field equations follow from the classical action by simple variation:

$$\delta S = \sum_{n=3}^{\infty} \frac{\kappa^{n-2}}{n!} n \{\delta\Psi, \Psi^{n-1}\}.$$

The gauge transformations of the theory are given by

$$\delta_\Lambda |\Psi \rangle = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} [\Psi^n, \Lambda].$$

Notice that ALL the terms of higher order are necessary for these to be consistent.

Similarly, in the sha context (cf. open string field theory as remarked below), one can define $N+1$ -point functions using the structure maps m_N . A particular example of such a structure which has quite recently appeared in the physics literature [W] can be expressed in terms of Massey products: Let M be a compact oriented manifold of dimension m and H its deRham cohomology. Poincaré duality gives a graded inner product on H and a pairing $\langle \cdot | \cdot \rangle$ between homology and cohomology. Massey products are determined by N -linear maps $H \otimes \cdots \otimes H \rightarrow H$ and $N+1$ -point functions then follow. Because the N -linear map has degree $2-N$ and the inner product is non-zero only for classes whose degrees sum to m , the only case in which all N -point functions $\langle yy \dots y \rangle$ have the same degree (and hence can appear simultaneously in a single Lagrangian) occurs when the degree of y is 1 and $m=2$. (A 3-fold Massey product in the complement of the Borromean rings detects the non-triviality of that link.) *With the grading conventions of closed string field theory, as in [Z], the non-polynomial Lagrangian has all fields Ψ of ghost degree 2.*

Fields: forms versus cohomology classes.

Recently the point of view of “cohomological physics” has become fairly common. Theories, both Lagrangian and Hamiltonian, are described initially in terms of a differential graded vector space or module (the differential is often referred to as a *BRST operator*), but the physical states are often cohomology classes, represented by closed “forms”. The structures we have been discussing, whether shLie or shassociative, may occur at either level [R], [HPT], [S3,5]. For example, closed string field theory [SZ], [KKS], [Z] exhibits the structure of an shLie algebra initially at the form level, but there is an induced structure on cohomology (where $d_1=0$, of course). With $d_1=0$, the bracket $[\cdot, \cdot]$ satisfies the Jacobi identity strictly, but this does NOT imply that the higher order brackets must be zero. In [WZ], Witten and Zwiebach describe the cohomology structure only, but there is one implicit at the form level as well, which Zwiebach works out very carefully in [Z].

In the open string field theory of [HIKKO], the structure is **homotopy associative** (the associating homotopy m_3 corresponds to the trilinear operation $(\circ \circ \circ)$ on 3 open string fields), but the higher order homotopies are zero. That is, HIKKO define a convolution product of string fields $\Phi * \Psi$ and establish the relation:

$$(\Phi * \Psi) * \Lambda - \Phi * (\Psi * \Lambda) = \pm Q(\Phi \circ \Psi \circ \Lambda) \pm (Q\Phi \circ \Psi \circ \Lambda) \pm (\Phi \circ Q\Psi \circ \Lambda) \pm (\Phi \circ \Psi \circ Q\Lambda),$$

but, proceeding to four fields, they find (with appropriate signs):

$$\Phi * (\Psi \circ \Lambda \circ \Sigma) \pm (\Phi \circ \Psi \circ \Lambda) * \Sigma \pm (\Phi * \Psi) \circ \Lambda \circ \Sigma \pm \Phi \circ (\Psi * \Lambda) \circ \Sigma \pm \Phi \circ \Psi \circ (\Lambda * \Sigma)$$

vanishes, rather than being equal to a BRST exact term, cf. (7). However, there are likely to be non-trivial higher order N -linear operations on the cohomology.

A reasonably general result that derives higher order operations on cohomology even when the underlying differential graded algebra is strictly associative or strictly Lie is given by Homological Perturbation Theory [HPT].

Theorem. *Let (V, d_1) be a differential graded vector space of the same homotopy type as a differential graded algebra (either associative or Lie) (A, d_A) . A specific homotopy equivalence induces the structure of an sh (respectively associative or Lie) algebra on V .*

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