N-differential graded algebras: examples and applications

Rafael Díaz

Universidad del Rosario

Bogotá, Colombia

The ideas presented in this talk are developed in the papers:

- M. Angel, R. Díaz, N-differential graded algebras, J. Pure App. Alg. 210 (2007) 673-683.
- M. Angel, R. Díaz, N-flat connections, in S. Paycha, B. Uribe (Eds.), Geometric and Topological Methods for Quantum Field Theory, Contemp. Math. 432, Amer. Math. Soc., Providence, pp. 1-36, 2007.
- M. Angel, Díaz, On the (3,N) Maurer-Cartan equation, J. Nonlinear Math. Phys. 14 (2007) 440-461.
- M. Angel, R. Díaz, A_∞^N-algebras, preprint, math.QA/0612661.

Introduction

The goal of this talk is to introduce the notion of

N-differential graded algebras.

According to Kapranov and Mayer a N-complex over a field k is a

$\mathbb{Z}$-graded k-vector space

$$V = \bigoplus_{n \in \mathbb{Z}} V_n$$

together with a degree one linear map

$d : V \longrightarrow V$ such that $d^N = 0$.

There are two generalizations of the notion of differential graded algebras to the context of N-complexes.

A choice, introduced by Kerner and further studied by Dubois-Violette and Kapranov, begins by fixing a primitive N-th root of unity q.

A q-differential graded algebra A is a $\mathbb{Z}$-graded associative algebra together with a degree one linear operator

$$d : A \longrightarrow A \quad \text{such that}$$

$$d(ab) = d(a)b + q^{\bar{a}} a d(b) \quad \text{and} \quad d^N = 0.$$

There are many interesting examples and constructions of q-differential graded algebras.

In this talk we consider another choice, the notion of

N-differential graded algebras.

A N-differential graded algebra (N-dga) over a field k is a triple

$$(A, m, d)$$

where

$$m : A \otimes A \longrightarrow A \quad \text{and} \quad d : A \longrightarrow A$$

are linear maps such that:

1. $d^N = 0$, that is (A, d) is a N-complex.
2. (A, m) is a graded associative algebra.
3. d satisfies the graded Leibnitz rule $d(ab) = d(a)b + (-1)^{\bar{a}} ad(b)$.

Categorical justification of the notion of N-dga.:

Let Nil-dgvect be the category of nilpotent complexes.

A nilpotent differential graded vector spaces is a pair (V, d) such that (V, d) is a N-complex for some integer $N \geq 1$.

Similarly we define the category Nil-dga of nilpotent differential graded algebras.

Objects in Nil-dga are pairs (A, d) such that (A, d) is a N-dga for some integer $N \geq 1$.

We have the following result:

- The category Nil-dgvect is a symmetric monoidal category.

  Indeed if (V, d_V) is a N-complex and (W, d_W) is a M-complex then
  $$(V \otimes W, d_V \otimes Id + Id \otimes d_W)$$
  is a $N+M-1$-complex.

- Nil-dga is the category of monoids in Nil-dgvect.

- Nil-dga inherits a symmetric monoidal structure from Nil-dgvetc.

In the rest of the talk I want to address the following issues:

- Examples of N-differential graded algebras.
- Deformations of N-differential graded algebras.
- N Lie algebroids.
- A_∞^N-algebras.

Examples of N-differential graded algebras.

The simplest construction of N-dga: N-flat connections.

Let $M \times V$ be a trivial vector bundle over M.

A connection

$$\omega \in \Omega^1(M, End(V)) \text{ on } M \times V \text{ is an}$$

$End(V)$-valued 1-form on M.

The connection ω defines a covariant derivative

$$d + \omega : \Omega(M, V) \rightarrow \Omega(M, V)$$

Proposition: $(\Omega(M,V), d+\omega)$ is a $2N$-differential graded algebra if and only if ω is a N-flat connection, that is the curvature

$$F = d\omega + \omega \wedge \omega$$

of ω satisfies the identity

$$F^N = 0.$$

Example. Consider $\mathbb{R}^4$ with coordinates (x_1, x_2, x_3, x_4).

Given smooth functions $\omega_1, \omega_2 : \mathbb{R}^2 \to \mathbb{R}$ consider the connection

$$\omega = \omega_1 dx_1 + \omega_2 dx_2.$$

A simple calculation shows that

$$F = \left(\frac{\partial \omega_1}{\partial x_2} - \frac{\partial \omega_2}{\partial x_1}\right) dx_1 \wedge dx_2 \neq 0, \quad \text{if} \quad \frac{\partial \omega_1}{\partial x_2} - \frac{\partial \omega_2}{\partial x_1} \neq 0.$$

For example one can take

$$\omega_1 = x_2 \text{ and } \omega_2 = -x_1,$$

so we have

$$F = 2dx_1 \wedge dx_2 \neq 0.$$

Moreover it is clear that

$$F^2 = 0.$$

Thus ω is a 2-flat connection and therefore

$$(\Omega(\mathbb{R}^4), d + \omega) \text{ is a } 4\text{-dga}.$$

Another geometric source of $2N$-differential graded algebras are certain kind of Riemannian manifolds.

Let (M, g) be a Riemannian manifold with associated covariant derivative ∇.

Assume that the tangent space TM of M admits an orthogonal decomposition

$$TM = A \oplus B$$

where

$$g \quad \text{is flat on} \quad B$$

then

$$(\Omega(M, TM), \nabla) \;\; \text{is a} \;\; 2N\text{-dga for} \;\; N \geq dim(A).$$

Differential forms of depth N on $\mathbb{R}^n$.

Fix an integer $N \geq 3$.

We construct the $(n(N-1)+1)$-differential graded algebra $\Omega_N(\mathbb{R}^n)$ of algebraic differential forms of depth N on $\mathbb{R}^n$.

Let $x_1, ..., x_n$ be coordinates on $\mathbb{R}^n$.

For $0 \leq i \leq n$ and $0 \leq j < N$ let $d^j x_i$ be a variable of degree j.

We identify $d^0 x_i$ with x_i.

The $(n(N-1)+1)$-differential graded algebra $\Omega_N(\mathbb{R}^n)$ is given by

$\Omega_N(\mathbb{R}^n) = \mathbb{R}[d^j x_i] / \left\langle d^j x_i d^k x_i \mid j,k \geq 1 \right\rangle$ as a graded algebras.

The $(n(N-1)+1)$-differential $d : \Omega_N(\mathbb{R}^n) \longrightarrow \Omega_N(\mathbb{R}^n)$ is given by

$$d(d^j x_i) = d^{j+1} x_i \quad \text{for} \quad 0 \leq j \leq N \quad \text{and} \quad d(d^{N-1} x_i) = 0.$$

One can show that d is $(n(N-1)+1)$-differential as follows:

It is easy to check that $\Omega_N(\mathbb{R})$ is a N-dga.

If A is a N-dga and B is a P-dga, then $A \otimes B$ is a $(N+P-1)$-dga.

$\Omega_N(\mathbb{R}^n) = \Omega_N(\mathbb{R})^{\otimes n}$.

The construction above can be generalized to define the algebra

$$\Omega_N(M)$$

of differential forms of depth N on each affine manifold.

Using a construction similar to Sullivan's construction of algebraic differential forms on simplicial sets, one can construct for each simplicial set S a nilpotent differential graded algebra

$$\Omega_N(S)$$

of differential forms of depth N on S.

Difference forms of degree N on affine discrete spaces.

There is a discrete analogue of the construction above.

We define the depth N analogue of Zeilberger's difference forms on discrete affine space.

Let $F(\mathbb{Z}^n, \mathbb{R})$ be the set of $\mathbb{R}$-valued maps on the lattice $\mathbb{Z}^n$.

Introduce variables $\delta^j m_i$ of degree j for $1 \leq i \leq n$ and $1 \leq j < N$.

The graded algebra of difference forms of depth N on $\mathbb{Z}^n$ is given by

$$D_N(\mathbb{Z}^n) = F(\mathbb{Z}^n, \mathbb{R}) \otimes \mathbb{R}[\delta^j m_i] / \left\langle \delta^j m_i \delta^k m_i \mid j, k \geq 1 \right\rangle .$$

A form $\omega \in D_N(\mathbb{Z}^n)$ can be written as

$$\omega = \sum_I \omega_I dm_I$$

where

$$I : \{1, .., n\} \longrightarrow \mathbb{N}$$

is any map and

$$dm_I = \prod_{i=1}^{n} d^{I(i)} m_i.$$

The degree of dm_I is

$$|I| = \sum_{i=1}^{n} I(i).$$

The finite difference $\Delta_i(g)$ of $g \in F(\mathbb{Z}^n, \mathbb{R})$ in the i-direction is given by

$$\Delta_i(g)(m) = g(m + e_i) - g(m),$$

where the vectors e_i are the canonical generators of $\mathbb{Z}^n$ and $m \in \mathbb{Z}^n$.

The difference operator δ is defined for $1 \leq j \leq N-2$ by the rules

$$\delta(g) = \sum_{i=1}^{n} \Delta_i(g)\delta m_i, \;\; \delta(\delta^j m_i) = \delta^{j+1} m_i \;\text{ and }\; \delta(\delta^{N-1} m_i) = 0.$$

It is not difficult to check that if $\omega = \sum_I \omega_I dm_I$, then

$$\delta\omega = \sum_J (\delta\omega)_J dm_J$$

where

$$(\delta\omega)_J = \sum_{J(i)=1} (-1)^{|J_{<i}|} \Delta_i \omega_{J-e_i} + \sum_{J(i)\geq 2} (-1)^{|J_{<i}|} \omega_{J-e_i}.$$

From the later formula we see that $(\delta\omega)_J$ is a linear combination of (differences of) functions ω_K with $|K| < |J|$. This fact implies that δ is nilpotent, indeed, one can check that $\delta^{n(N-1)+1} = 0$.

All together we have the following result.

$D_N(\mathbb{Z}^n)$ is a graded algebra and the difference operator δ gives $D_N(\mathbb{Z}^n)$ the structure of a $(n(N-1)+1)$-complex.

One can check that δ satisfies a twisted Leibnitz rule, so

$$D_N(\mathbb{Z}^n)$$

is actually pretty close of being a N-dga.

Again using a Sullivan's construction one associated an algebra of difference forms of depth N to each simplicial set. The algebra thus obtained is a (twisted) nilpotent-differential graded algebra.

Next we consider the deformation theory of nilpotent-differential graded algebras.

The (N, M) Maurer-Cartan equation controls the deformations of a N-dga into a M-dga.

Let us review the construction of (N, M) Maurer-Cartan equation.

Let (A, d) be a N-differential graded algebra and e a degree one derivation of A.

The (N, M) Maurer-Cartan equation determines under which conditions $(A, d + e)$ ia a M-differential graded algebra.

For $s = (s_1, ..., s_n) \in \mathbb{N}^n$ we set $l(s) = n$ and $|s| = \sum_i s_i$.

For $1 \leq i < n$ we set $s_{>i} = (s_{i+1}, ..., s_n)$.

For $1 < i \leq n$ we set $s_{<i} = (s_1, ..., s_{i-1})$ and $s_{>n} = s_{<1} = \emptyset$.

$\mathbb{N}^{(\infty)}$ denotes the set $\bigsqcup_{n=0}^{\infty} \mathbb{N}^n$ where by convention $\mathbb{N}^0 = \{\emptyset\}$.

For any $c : A \longrightarrow A$ set

$$\widehat{d}(c) = [d, c].$$

For $s \in \mathbb{N}^{(\infty)}$ we set

$$e^{(s)} = e^{(s_1)} ... e^{(s_n)}$$

where

$$e^{(a)} = \widehat{d}^a(e) \quad \text{if} \quad a \geq 1$$

$$e^{(0)} = e \quad \text{and} \quad e^{\emptyset} = 1.$$

For $M \in \mathbb{N}$ we let

$$E_M = \{s \in \mathbb{N}^{(\infty)} : |s| + l(s) \leq M\}$$

and for $s \in E_M$ we define the integer $M(s)$ by

$$M(s) = M - |s| - l(s).$$

Recall that one can associated a discrete quantum mechanical system to each directed graph together with a weight attached to each of its edges.

Let us introduce a discrete quantum mechanical system given by the following data:
1. The set of vertices is $\mathbb{N}^{(\infty)}$.

2. There is a unique directed edge from vertex s to vertex t if and only if

$$t \in \{(0,s)\} \cup \{s\} \cup \{s + e_i \mid 1 \leq i \leq l(s)\}$$

where $e_i = (0, .., \underbrace{1}_{i-th}, .., 0) \in \mathbb{N}^{l(s)}$.

3. An edge e with source $s(e)$ and target $t(e)$ is weighted according to the table

$s(e)$	$t(e)$	$v(e_i)$
s	$(0, s)$	1
s	s	$(-1)^{\|s\|+l(s)}$
s	$s+e_i$	$(-1)^{\|s_{<i}\|+i-1}$

The set $P_M(\emptyset, s)$ consists of all paths

$$\gamma = (e_1, ..., e_M)$$

such that

$$s(e_1) = \emptyset \quad \text{and} \quad t(e_M) = s.$$

The weight $v(\gamma)$ of $\gamma \in P_M(\emptyset, s)$ is $v(\gamma) = \prod_{i=1}^{M} v(e_i)$.

The (N, M) Maurer-Cartan equation is given by

$$\sum_{k=1}^{M-1} c_k d^k = 0$$

where

$$c_k = \sum_{\substack{s \in E_M \\ M(s)=k \\ s_i < N}} c(s, M) e^{(s)} \quad \text{and} \quad c(s, M) = \sum_{\gamma \in P_M(\emptyset, s)} v(\gamma).$$

For example we have that

If (A,d) is a 3-dga then $(A,d+e)$ is a 3-dga if and only if

$$(d^2(e)+d(e)e+e^3)+(d(e)+e^2)d+ed^2=0.$$

If (A,d) is a 3-dga then $(A,d+e)$ is a 4-dga if and only if

$$(e^4+e^2d(e)+d(e)e^2+d^2(e)e+ed^2(e)+(d(e))^2)+2(e^2+d(e))d^2=0.$$

In deformation theory one also considers "infinitesimal"deformations

that is deformations of the form $d+te$

where t is a formal variable such that $t\neq 0$ but $t^2=0$.

We set

$$Par(k, N-k+1) = \{p = (p_1, \cdots, p_{N-k+1}) \mid \sum_{i=1}^{N-k+1} p_i = k\}$$

$$w(p) = \sum_{i=1}^{N-k+1} (i-1)p_i.$$

Let (A, d) be a N-dga then e defines an infinitesimal deformation of the N-dga into the N-dga $(A[[t]]/(t^2), d + te)$ if and only if

$$\sum_{k=0}^{N-1} \left(\sum_{p \in Par(k, N-k+1)} (-1)^{w(p)} \right) \widehat{d}^{N-k-1}(e) d^{N-k-1} = 0.$$

N Lie algebroids.

We recall the definition of a Lie algebroid.

Let E be a finite dimensional vector bundle. E is a Lie algebroid if and only if $\Gamma(\bigwedge E^*)$ is a differential graded algebra. A differential on $\bigwedge E^*$ is the same as a degree one vector field v on $E[-1]$ such that $v^2 = 0$.

Weakening the conditions on the vector field v one gets various generalizations of the notion of Lie algebroids.

If one let v of arbitrary degree, one get the notion of L_∞ algebroid.

If instead of $v^2 = 0$ on impose higher order nilpotency we get the notion of N Lie algebroids.

Higher nilpotency together with arbitrary degree yields the notion of N L_∞ algebroids.

N L_∞ algebroids have not been deeply studied so far.

An important subtlety is that the right higher nilpotency condition is not the naive $v \circ .... \circ v = 0$ as one could imagine.

Define the non-associative product on vector fields given by

$$v_i \partial_i \diamond w_j \partial_j = (v_i \partial_i w_j) \partial_j .$$

A N Lie algebroid is a vector bundle E together with a degree one derivation $d : \Gamma(\bigwedge E^*) \longrightarrow \Gamma(\bigwedge E^*)$, such that $d \diamond d \diamond \cdots \diamond d = 0$.

It is not difficult to find examples of N Lie algebroids. The simplest examples are the N Lie algebras.

A N Lie algebra is a vector space $\mathfrak{g}$ together with a degree one derivation d on $\bigwedge \mathfrak{g}^*$ such that the N-th $\diamond$-composition of d with itself vanishes.

We describe 3 Lie algebras in more familiar terms.

For integers $k_1, k_2, ..., k_l$ such that $k_1 + k_2 + \cdots + k_l = n$, we let $Sh(k_1, k_2, \cdots, k_l)$ be the set of permutations

$$\sigma : \{1, \cdots, n\} \longrightarrow \{1, \cdots, n\}$$

such that σ is increasing on the intervals $[k_i + 1, k_{i+1}]$ for $0 \leq i \leq l$, $k_0 = 1$ and $k_{l+1} = n$.

Assume we are given a map $[\ ,\ ] : \bigwedge^2 \mathfrak{g} \longrightarrow \mathfrak{g}$.

The pair $(\mathfrak{g}, [\ ,\ ])$ is a 3 Lie algebra if and only if for $v_1, v_2, v_3, v_4 \in \mathfrak{g}$ we have

$$\sum_{\sigma \in Sh(2,1,1)} sgn(\sigma)[[[v_{\sigma(1)}, v_{\sigma(2)}], v_{\sigma(3)}]v_{\sigma(4)}] =$$

$$= \sum_{\sigma \in Sh(2,2)} sgn(\sigma)[[v_{\sigma(1)}, v_{\sigma(2)}], [v_{\sigma(3)}, v_{\sigma(4)}]]$$

We closed this talk mentioning that there is a higher depth analogue of the notion of N infinity algebras which we called

$$A_\infty^N - \text{algebras.}$$

An A_∞^N-algebra is a graded vector space A together with a sequence of degree one maps

$$m_k : A[1]^{\otimes k} \longrightarrow A[1]$$

such that the associated coderivation $\delta = \sum m_i$ on $T(A[1])$ satisfies

$$\delta \diamond \delta .... \diamond \delta = 0.$$

The condition $\delta \diamond \delta = 0$ defining A^2_∞-algebras is the familiar condition for A_∞-algebras:

$$\sum_{r+s+t=n} m_{r+1+t} \circ (1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0.$$

The condition $\delta \diamond \delta \diamond \delta = 0$ defining an A^3_∞-algebra is

$$\sum_{a+b+c+d+e=n} m_{a+e+1} \circ (1^{\otimes a} \otimes m_{b+d+1} \otimes 1^{\otimes e}) \circ (1^{\otimes a+b} \otimes m_c \otimes 1^{\otimes d+e}) \quad + \\ m_{a+b+c+e+1} \circ (1^{\otimes a} \otimes m_b \otimes 1^{\otimes c+e+1}) \circ (1^{\otimes a+b+c} \otimes m_d \otimes 1^{\otimes e}) \quad + \\ m_{a+c+d+e+1} \circ (1^{\otimes a+c+1} \otimes m_d \otimes 1^{\otimes e}) \circ (1^{\otimes a} \otimes m_b \otimes 1^{\otimes c+d+e}) \quad = \quad 0$$

It becomes rather cumbersome to write an explicit formula for the condition $\delta \diamond \delta .... \diamond \delta = 0$ for $N \geq 4$.

We rewrite that condition in terms of trees.

Let RT_l^n be the set of isomorphisms classes of rooted planar trees with l leaves and n internal vertices.

For example the following trees are in RT_{16}^3:

The condition $\delta \diamond \delta .... \diamond \delta = 0$ holds if and only if for each $l \in \mathbb{N}_+$

$$\sum_{\Gamma \in RT_l^N} m_\Gamma = 0,$$

where m_Γ is defined by a procedure similar to that use in the graphical definition of A_∞-algebras putting the m_s operator on each vertex with s incoming edges attached to it.

¡THANKS!