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AN OPERAD ACTION ON INFINITE LOOP SPACE MULTIPLICATION

THOMAS LADA

It is well-known that an infinite loop space is an H-space whose multiplication enjoys nice properties concerning associativity and commutativity. A practical way of identifying infinite loop spaces is the utilization of May's recognition principle [**3**; **4**]. To apply this principle, one requires an E_∞-operad action on a space X; this action gives rise to various multiplications on X. In this note, it is shown that such multiplications enjoy an operad action up to homotopy that encodes the associativity and commutativity information, and that May's delooping theorem may be applied to them. We refer to [**3**] for the terminology of operads and monads.

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To be more precise, let $\{\mathscr{C}(j)\}$ be the infinite little cube operad of Boardman and Vogt [**1**] as described in [**3**, Chapter 5], and let (C, μ, η) be the induced monad. Let (X, θ) be an infinite loop space where $\theta : CX \to X$ is a C-space structure map; this map may be viewed as a collection of equivariant maps $\theta_n : \mathscr{C}(n) \times X^n \to X$ that commute with the internal operad structure [**3**, Lemma 1.4]. The space X^2 is also a C-space with $(\theta \times \theta)_n : \mathscr{C}(n) \times (X^2)^n \to X^2$ defined by $(\theta \times \theta)_n(d; x_1, y_1, \ldots, x_n, y_n) = (\theta_n(d; x_1, \ldots, x_n), \theta_n(d; y_1, \ldots, y_n))$. If $c \in \mathscr{C}(2)$ is any point, then the operad action on X restricts to yield a map $\theta_2(c) : X^2 \to X$, i.e. a multiplication on X.

Our main result is

THEOREM 1. *$\theta_2(c) : X^2 \to X$ is a strong homotopy C-map for any choice of $c \in \mathscr{C}(2)$.*

Let us recall the definition of s.h.C-maps as well as some of their properties.

Definition 2. Let (X, ξ) and (Y, ϕ) be C-spaces. Then $f : X \to Y$ is an s.h.C-map if there exists a collection of homotopies $\{f_n\}$ with

$$f_n : I^n \times C^n X \to Y$$

satisfying

$$f_n(t_1, \ldots, t_n, z) = \begin{cases} \phi \circ Cf_{n-1}(t_2, \ldots, t_n, z) & \text{if } t_1 = 0 \\ f_{n-1}(t_1, \ldots, \hat{t}_j, \ldots, t_n, C^{j-2}\mu_{n-j}(z)) & \text{if } t_j = 0 \\ f \circ \xi_{n-1}(t_2, \ldots, t_n, z) & \text{if } t_1 = 1 \\ f_{j-1}(t_1, \ldots, t_{j-1}, C^{j-1}\xi_{n-j}(t_{j+1}, \ldots, t_n, z)) & \text{if } t_j = 1. \end{cases}$$

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Here, μ_i is the natural transformation $\mu C^i : C^2C^i \to CC^i$. In [**2**, Chap. V, Cor. 5.2, Prop. 2.5] it is shown that if $f : X \to Y$ is an s.h.C-map, then there exist C-spaces UX and UY containing X and Y respectively as deformation retracts, and a C-map $Uf : UX \to UY$ such that the diagram

$$\begin{array}{ccc} UX & \xrightarrow{Uf} & UY \\ \eta \big\uparrow & & \big\uparrow \eta \\ X & \xrightarrow{f} & Y \end{array}$$

commutes as a diagram of s.h.C-maps. As defined in [**2**, Construction 2.1],

$$UX = \coprod_q I^q \times C^{q+1}X/\sim$$

where

$$(t_1, \ldots, t_q, x) \sim \begin{cases} (t_1, \ldots, \hat{t}_j, \ldots, t_n, C^{j-1}\mu_{q-j}(x)) & \text{if } t_j = 0 \\ (t_1, \ldots, t_{j-1}, C^j\xi_{q-j}(t_{j+1}, \ldots, t_q, x)) & \text{if } t_j = 1, \end{cases}$$

and $\eta : X \to CX$ is defined by $\eta(x) = (1, x) \in \mathscr{C}(1) \times X$ where $1 \in \mathscr{C}(1)$ is the point that acts as identity in the operad structure [**3**, Def. 1.1]. Of course, May's delooping theorem may now be applied to the C-map Uf.

It should be noted that the natural examples of C-spaces are infinite loop spaces, and that their multiplications are clearly infinite loop maps by the additivity of the stable category. I originally hoped to prove Theorem 1 for arbitrary E_∞ operads and not just $\mathscr{C}$, but I was unable to find a proof that did not depend on the geometry of the little cubes. Nevertheless, the methods here may be of interest as presenting a model of what actually is involved in the verification that a particular map is an s.h.C-map, thus providing an illustrative example for the general theory of [**2**, Chap. V].

To begin, let us fix a point $c \in \mathscr{C}(2)$ and define a map $\alpha_0 : X^2 \to CX$ via $\alpha_0(x, y) = (c, x, y) \in \mathscr{C}(2) \times X^2$. Note that the composition $\theta \circ \alpha_0 : X^2 \to X$ is equal to the map $\theta_2(c, x, y)$.

Now define a map $\alpha : UX^2 \to CUX$ where $\alpha : I^n \times C^{n+1}X^2 \to I^n \times C^{n+2}X$ is given by

$$\alpha : I^n \times \mathscr{C}^M \times (X^2)^N \to I^n \times \mathscr{C}(2) \times (\mathscr{C}^M)^2 \times X^N \times Y^N$$

such that $\alpha(t_1, \ldots, t_n, z, x_1, y_1, \ldots, x_N, y_N) = (t_1, \ldots, t_n, c, z, z, x_1, \ldots, x_N, y_1, \ldots, y_N)$ where X^N and Y^N are the first and second coordinates of $(X^2)^N$ respectively, and M and N are integers large enough to make sense. We will require

LEMMA 3. *α is well-defined.*

and

LEMMA 4. *The composition $\mu \circ \alpha : UX^2 \to UX$ is an s.h.C-map.*

In Lemma 4, $\mu : CUX \to UX$ is the natural C-space structure map defined in [**2**, Chap. V, Cor. 2.4] induced by the natural transformation $\mu : C^2 \to C$.

Proof of Theorem 1. Consider the diagram

$$\begin{array}{ccccc} UX^2 & \xrightarrow{\alpha} & CUX & \xrightarrow{\mu} & UX \\ \uparrow{\scriptstyle \eta} & & & & \downarrow{\scriptstyle r} \\ X^2 & \xrightarrow{\alpha_0} & CX & \xrightarrow{\theta} & X \end{array}$$

where r is the retraction $UX \to X$ defined by $r(t_1, \ldots, t_n, z) = \theta_n(t_1, \ldots, t_n, z)$ where $\theta_n : I^n \times C^{n+1}X \to X$. This diagram commutes since $r \circ \mu \circ \alpha \circ \eta(x, y) = r \circ \mu \circ \alpha(1, x, y) = r \circ \mu(c, 1, 1, x, y) = r(c, x, y) = \theta_2(c, x, y)$. Moreover, η is an s.h.C-map, r is a C-map and thus the composition $r \circ \mu \circ \alpha \circ \eta$ is an s.h.C-map.

Proof of Lemma 3. Since α is defined on the operad level, it must be verified that α respects the relation defining the functor C defined in [**3**, p. 13]. We use induction and first show that $\alpha : CX^2 \to C^2X$ is well-defined:

1) Equivariance: we have $\alpha : \mathscr{C}(k) \times (X^2)^k \times \mathscr{C}(2) \times \mathscr{C}(k)^2 \times X^{2k}$; let $(d, x_1, y_1, \ldots, x_k, y_k) \in \mathscr{C}(k) \times (X^2)^k$ and $\sigma \in \Sigma_k$. Then

$$(d\sigma, x_1, y_1, \ldots, x_k, y_k) \sim (d, x_{\sigma^{-1}(1)}, y_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(k)}, y_{\sigma^{-1}(k)})$$

but

$$\begin{aligned} \alpha(d\sigma, x_1, y_1, \ldots, x_k, y_k) &= (c, d\sigma, d\sigma, x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k) \\ &\sim (c, d, d, x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(k)}, y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(k)}) \\ &= \alpha(d, x_{\sigma^{-1}(1)}, y_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(k)}, y_{\sigma^{-1}(k)}). \end{aligned}$$

2) Base point identifications: suppose $(x_i, y_i) = (*, *)$ where $*$ is the base point of X. Then if we write $d = (d_1, \ldots, d_k) \in \mathscr{C}(k)$, we have

$$\begin{aligned} &(d_1, \ldots, d_k, x_1, y_1, \ldots, x_k, y_k) \\ &\qquad \sim (d_1, \ldots, \hat{d}_i, \ldots, d_k, x_1, y_1, \ldots, \hat{x}_i, \hat{y}_i, \ldots, x_k, y_k). \end{aligned}$$

But

$$\begin{aligned} &\alpha(d_1, \ldots, d_k, x_1, y_1, \ldots, x_k, y_k) \\ &\qquad = (c, d_1, \ldots, d_k, d_1, \ldots, d_k, x_1, \ldots, x_k, y_1, \ldots, y_k) \\ &\sim (c, d_1, \ldots, \hat{d}_i, \ldots, d_k, d_1, \ldots, \hat{d}_i, \ldots, d_k, x_1, \ldots, \hat{x}_i, \ldots, x_k, y_1, \ldots, \hat{y}_i, \ldots, y_k) \\ &= \alpha(d_1, \ldots, \hat{d}_i, \ldots, d_k, x_1, y_1, \ldots, \hat{x}_i, \hat{y}_i, \ldots, x_k, y_k). \end{aligned}$$

In 1) and 2) the relations denoted by $\sim$ are those used in the construction of a monad from an operad referred to above. Now assume the map $\alpha : C^{n-1}X^2 \to C^nX$

is well-defined. A straightforward calculation identical to the one above allows us to conclude that $\alpha : C^n X^2 \to C^{n+1}X$ is well-defined.

We now demonstrate that α respects the relation defining UX^2. We have

$$\alpha_n : I^n \times C^{n+1}X^2 \to I^n \times C^{n+2}X.$$

If $t_j = 0$, $(t_1, \ldots, t_n, z) \sim (t_1, \ldots, \hat{t}_j, \ldots, t_n, C^{j-1}\mu_{n-j}(z))$. For α to be well-defined, it is necessary that

$$\alpha \circ C^{j-1}\mu_{n-j} = C^j\mu_{n-j} \circ \alpha.$$

To see this, on the operad level the maps are defined from

$$\mathscr{C}^N \times \mathscr{C}(k) \times \mathscr{C}(j_1) \times \ldots \times \mathscr{C}(j_k) \times \mathscr{C}^M \times (X^2)^L$$

to

$$\mathscr{C}(2) \times (\mathscr{C}^N)^2 \times \mathscr{C}(j)^2 \times (\mathscr{C}^M)^2 \times X^{2L}$$

where $j = j_1 + \ldots + j_k$. By writing down the appropriate commutative diagram, it is easy to see that the above equality is true.

On the other hand, if $t_j = 1$,

$$(t_1, \ldots, t_n, z) \sim (t_1, \ldots, t_{j-1}, C^j(\theta \times \theta)_{n-j}(t_{i+1}, \ldots, t_n, z)).$$

We have to show that

$$\alpha \circ C^j(\theta \times \theta)_{n-j} = C^{n+1}(\theta)_{n-j} \circ \alpha.$$

Recall that $(\theta \times \theta)_{n-j} = (\theta \times \theta) \circ \mu \circ \mu_1 \circ \ldots \circ \mu_{n-j-1}$ and that $\theta_{n-j} = \theta \circ \mu \circ \mu_1 \circ \ldots \circ \mu_{n-j-1}$. Also,

$$\begin{aligned} C^{j+1}\theta_{n-j} \circ \alpha &= C^{j+1}\theta \circ C^{j+1}\mu \circ \ldots \circ C^{j+1}\mu_{n-j-1} \circ \alpha \\ &= C^{j+1}\theta \circ C^{j+1}\mu \circ \ldots \circ \alpha \circ C^j\mu_{n-j-1} \\ &= \text{-------------------------} \\ &= C^{j+1}\theta \circ \alpha \circ C^j\mu \circ \ldots \circ C^j\mu_{n-j-1}. \end{aligned}$$

Since $\alpha \circ C^j(\theta \times \theta)_{n-j} = \alpha \circ C^j(\theta \times \theta) \circ C^j\mu \circ \ldots \circ C^j\mu_{n-j-1}$, we have to show that $C^{j+1}\theta \circ \alpha = \alpha \circ C^j(\theta \times \theta)$. Let us choose a point

$$[d, e_1, \ldots, e_k, (x_1, y_1, \ldots, x_{j_1}, y_{j_1}), \ldots, (x_K, y_K, \ldots, x_{K+j_k}, y_{K+j_k})] \in C^{j+1}X^2$$

where $K = \sum_{i=1}^{k=1} j_i$. Apply α to get

$$[c, d^2, e_1{}^2, \ldots, e_k{}^2, x_1, \ldots, x_{K+j_k}, y_1, \ldots, y_{K+j_k}]$$

and then apply $C^{j+1}\theta$ to get

$$[c, d^2, \theta_{j_1}(e_1, x_1, \ldots, x_{j_1}), \ldots, \theta_{j_k}(x_K, \ldots, x_{K+j_k}), \theta_{j_1}(e_1, y_1, \ldots, y_{j_1}), \ldots].$$

Now apply $C^j(\theta \times \theta)$ to the point chosen above to get

$$[d, \theta_{j_1}(e_1, x_1, \ldots, x_{j_1}), \theta_{j_1}(e_1, y_1, \ldots, y_{j_k}), \ldots, \theta_{j_k}(e_k, x_K, \ldots, x_{K+j_k}), \ldots]$$

and then apply α to get

$$[c, d^2, \theta_{j_1}(e_1, x_1, \ldots, x_{j_1}), \ldots, \theta_{j_k}(x_K, \ldots, x_{K+j_k}), \theta_{j_1}(e_1, y_1, \ldots, y_{j_1}), \ldots].$$

Proof of Lemma 4. To show that $\mu \circ \alpha : UX^2 \to UX$ is an s.h.C-map, we will construct a family of homotopies $h_n : I^n \times C^n UX \to UX$ which satisfy the requisite conditions of Definition 2. We will describe these homotopies on the operad level and take care that they will be compatible with the relations defining C and U. In order to proceed with the construction of h_n, we require the utilization of a specific element $c \in \mathscr{C}(2)$. Recall that $c = \langle c_1, c_2 \rangle$ where each $c_i : (I^o)^n \to (I^o)^n$ is a linear embedding with parallel axes; also, $\operatorname{im}(c_1) \cap \operatorname{im}(c_2) = \emptyset$. Let us choose c_1 to be the linear embedding defined by $y = \frac{1}{2}x$ in the first coordinate and the identity map in the remaining coordinates; similarly, we choose c_2 to be defined by $y = \frac{1}{2}x + \frac{1}{2}$ in the first coordinate and the identity map in the remaining coordinates.

We first construct $h_1 : I \times CUX^2 \to UX$ such that $h_1|_0 = \mu \circ C\mu \circ C\alpha$ and $h_1|_1 = \mu \circ \alpha \circ \mu$; on the operad level, h_1 is a map

$$h_1 : I \times I^n \times \mathscr{C}(k) \times \mathscr{C}(j_1) \times \ldots \times \mathscr{C}(j_k) \times \mathscr{C}^M \times (X^2)^N \to I^n \times \mathscr{C}(\textstyle\sum 2j_i) \times \mathscr{C}^{2M} \times X^{2N}.$$

Let us choose a point in $I^n \times \mathscr{C}(k) \times \mathscr{C}(j_1) \times \ldots \times \mathscr{C}(j_k) \times \mathscr{C}^M \times (X^2)^N$, say

$$(t_1, \ldots, t_n, d, e_1, \ldots, e_k, z, x_1, y_1, \ldots, x_N, y_N).$$

We require that

$$\begin{aligned} h_1|_0 &= \mu \circ C\mu \circ C\alpha(t_1, \ldots, t_n, d, e_1, \ldots, e_k, z, x_1, y_1, \ldots, x_N, y_N) \\ &= \mu \circ C\mu(t_1, \ldots, t_n, d, c, e_1{}^2, \ldots, c, e_k{}^2, z^2, x_1, \ldots, x_{Nj_1}, y_1, \ldots, y_{Nj_1}, \ldots) \\ &= \mu(t_1, \ldots, t_n, d, \gamma(c; e_1{}^2), \ldots, \gamma(c; e_k{}^2), z^2, \underline{\qquad\qquad\qquad\qquad}) \\ &= (t_1, \ldots, t_n, \gamma(\gamma(d; c^k; e_1{}^2, \ldots, e_k{}^2)), z^2, \underline{\qquad\qquad\qquad\qquad}). \end{aligned}$$

Here γ is the little cube operad "multiplication" that induces μ. On the other hand,

$$\begin{aligned} h_1|_1 &= \mu \circ \alpha \circ \mu(t_1, \ldots, t_n, d, e_1, \ldots, e_k, z, x_1, y_1, \ldots, x_N, y_N) \\ &= \mu \circ \alpha(t_1, \ldots, t_n, \gamma(d; e_1, \ldots, e_k), z, x_1, y_1, \ldots, x_N, y_N) \\ &= \mu(t_1, \ldots, t_n, c; \gamma(d; e_1, \ldots, e_k)^2, z^2, x_1, \ldots, x_N, y_1, \ldots, y_N) \\ &= (t_1, \ldots, t_n, \gamma(c; \gamma(d; e_1, \ldots, e_k)^2), z^2, x_1, \ldots, x_N, y_1, \ldots, y_N) \\ &= (t_1, \ldots, t_n, \gamma(\gamma(c; d^2), e_1, \ldots, e_k, e_1, \ldots, e_k), z^2, x_1, \ldots, x_N, y_1, \ldots, y_N) \\ &\sim (t_1, \ldots, t_n, \gamma(\gamma(c; d^2), e_1, \ldots, e_k, e_1, \ldots, e_k)\sigma, z^2\sigma, x_1, \ldots, x_{Nj_1}, \\ &\qquad\qquad y_1, \ldots, y_{Nj_1}, \ldots) \\ &= (t_1, \ldots, t_n, \gamma(\gamma(c; d^2)\sigma, e_1{}^2, \ldots, e_k{}^2), z^2\sigma, x_1, \ldots, x_{Nj_1}, y_1, \ldots, y_{Nj_1}, \ldots). \end{aligned}$$

The use of the $\sim$ two lines above is justified by the fact that since we want h_1 to be defined on the monad level, we may work with equivalent points on the operad level. The evident shuffle permutation above is denoted by σ.

It follows from the above calculation that we require a path in $\mathscr{C}(2k)$ between $\gamma(d; c^k)$ and $\gamma(c; d^2)\sigma$. A path between these points certainly exists since $\mathscr{C}(2k)$ is contractible by definition; however, we need to make certain that the path respects basepoint identifications. For this, we require the geometry of the little cubes. Let $d = \langle d_1, \ldots, d_k \rangle$ where the little cube d_i has first coordinate given by $y = (z_2{}^i - z_1{}^i)x + z_1{}^i$ where $0 \leqq z_1 < z_2 \leqq 1$. We have that

$$\gamma(c; d^2) = \langle c_1 \circ d_1, c_1 \circ d_2, \ldots, c_1 \circ d_k, c_2 \circ d_1, \ldots, c_2 \circ d_k \rangle$$

and

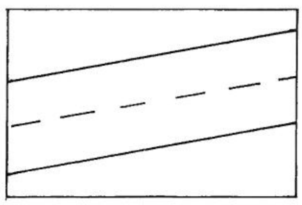
$$\begin{aligned} \gamma(d; c^k)\sigma &= \langle d_1 \circ c_1, d_1 \circ c_2, d_2 \circ c_1, d_2 \circ c_2, \ldots, d_k \circ c_1, d_k \circ c_2 \rangle \sigma \\ &= \langle d_1 \circ c_1, d_2 \circ c_1, \ldots, d_k \circ c_1, d_1 \circ c_2, d_2 \circ c_2, \ldots, d_k \circ c_2 \rangle. \end{aligned}$$

We thus need only to describe a path between $d_i \circ c_j$ and $c_j \circ d_i$. Let us examine the first coordinates of these cubes. We have

$$c_1 \circ d_i = \tfrac{1}{2}(z_2{}^i - z_1{}^i)x + \tfrac{1}{2}z_1{}^i \quad \text{and} \quad d_i \circ c_1 = \tfrac{1}{2}(z_2{}^i - z_1{}^i)x + z_1{}^i.$$

Geometrically, these are parallel lines and a path between them may be described by

$$f_1{}^i(t) = \tfrac{1}{2}\,(z_2{}^i - z_1{}^i)x + \frac{2-t}{2}\,z_1{}^i.$$



$\leftarrow f_1{}^i(t)$.

Similarly,

$$c_2 \circ d_i = \tfrac{1}{2}(z_2{}^i - z_1{}^i)x + \tfrac{1}{2}(z_2{}^i - z_1{}^i) + \tfrac{1}{2}$$

and

$$d_i \circ c_2 = \tfrac{1}{2}(z_2{}^i - z_1{}^i)x + \tfrac{1}{2}(z_2{}^i - z_1{}^i) + z_1{}^i$$

and a path between these parallel lines may be taken to be

$$g_1{}^i(t) = \tfrac{1}{2}(z_2{}^i - z_1{}^i)x + \tfrac{1}{2}t + (1-t)z_1{}^i.$$

We thus may define

$$h_1(t) = \langle f_1{}^1(t), g_1{}^1(t), \ldots, f_1{}^i(t), g_1{}^i(t), \ldots, f_1{}^k(t), g_1{}^k(t) \rangle.$$

As we have to glue together these homotopies with the relation defining a monad from an operad, we must now exercise a bit of caution. It is clear that this homotopy is equivariant as it is defined coordinate-wise. It is also clear that this homotopy respects basepoint relations as such identifications amount to deleting the appropriate coordinate. However, if we let K = geometrical dimension of the little cubes in question, our path may not remain in $\mathscr{C}_K(2k)$; i.e. any reasonable path such as the one given above depends on the number k of little cubes involved. To remedy this, we modify $h_1(t)$ slightly in the following fashion: let us first embed the point $\langle d_1 \circ c_1, d_1 \circ c_2, \ldots, d_k \circ c_1, d_k \circ c_2 \rangle$ in $\mathscr{C}_{K+1}(2k)$ via $\langle d_1 \circ c_1 \times 1, d_1 \circ c_2 \times 1, \ldots, d_k \circ c_1 \times 1, d_k \circ c_2 \times 1 \rangle$ where 1 is the identity map $I^o \to I^o$. Now describe our homotopy by shrinking each $d_i \circ c_1 \times 1$ to $d_i \circ c_1 \times (0, \frac{1}{2})$ and each $d_i \circ c_2 \times 1$ to $d_i \circ c_2 \times (\frac{1}{2}, 1)$. We may now translate each $d_i \circ c_1 \times (0, \frac{1}{2})$ to $c_1 \times d_i \circ (0, \frac{1}{2})$ by $f_1{}^i(t)$ and each $d_i \circ c_2 \times (\frac{1}{2}, 1)$ to $c_2 \circ d_i \circ (\frac{1}{2}, 1)$ by $g_1{}^i(t)$ as described above without possibility of collision. We then expand $c_1 \circ d_i \times (0, \frac{1}{2})$ to $c_1 \circ d_i \times (0, 1)$ and $c_2 \circ d_i \times (\frac{1}{2}, 1)$ to $c_2 \circ d_i \times (0, 1)$. This procedure makes certain that our path remains in $\mathscr{C}_\infty(2k)$. To facilitate notation and parametrization, we will not formally write down this expanding and shrinking; we will, however, assume that it has been done whenever necessary.

We now proceed to construct h_n inductively. Assuming that we have defined h_{n-1}, we exhibit h_n as an appropriate path from Ch_{n-1} to $\mu \circ \alpha \circ \mu_{n-1}$. Note that $\mu \circ \alpha \circ \mu_{n-1}$ applied to a point is an $n + 1$ fold composition of little cubes in each of the N coordinates; this composition is of the form $c_i \circ \lambda_1 \circ \lambda_2 \circ \ldots \circ \lambda_n$ with $i = 1, 2$. Let the first coordinate of each λ_i be given by the line $y = (z_2{}^i - z_1{}^i)x + z_1{}^i$. Again, our homotopy will involve a translation of the first coordinate of the cube in question. The first coordinate of $\lambda_1 \circ \ldots \circ \lambda_n$ is given by

$$y = \prod_{i=1}^{n} (z_2{}^i - z_1{}^i)x + \sum_{i=1}^{n} z_1{}^i \prod_{j=1}^{i-1} (z_2{}^j - z_1{}^j).$$

Again, we define a path for each coordinate and each c_i. Specificially, for c_1, we define

$$f_n(t_1, \ldots, t_n, \lambda_1, \ldots, \lambda_n) = \frac{1}{2} \prod_{i=1}^{n} (z_2{}^i - z_1{}^i)x + \sum_{i=1}^{n} \frac{1 + P(t_1, \ldots, t_i)}{2} z_1{}^i \prod_{j=1}^{i-1} (z_2{}^j - z_1{}^j)$$

where

$$P(t_1, \ldots, t_i) = \prod_{j=1}^{i} (1 - t_j)$$

and

$$\prod_{j=1}^{i-1} (z_2{}^j - z_1{}^j) = 1 \quad \text{if } i = 1.$$

On the other hand for c_2, we define

$$g_n(t_1, \ldots, t_n, \lambda_1, \ldots, \lambda_n) = f_n(t_1, \ldots, t_n)$$
$$+ \tfrac{1}{2}t_1 + (1 - t_1)(z_2^n - z_1^n)(g_{n-1}(t_2, \ldots, t_n) - f_{n-1}(t_2, \ldots, t_n)).$$

We then of course define

$$h_n = (f_n^1, g_n^1, \ldots, f_n^N, g_n^N).$$

To verify compatibility with h_j, $j < n$, we begin with $t_1 = 0$. Then

$$f_n(t_1, \ldots, t_n) = \frac{1}{2} \prod_{i=1}^{n} (z_2^i - z_1^i)x + \sum_{i=2}^{n} \frac{1 + P(t_2, \ldots, t_i)}{2}$$
$$\times z_1^i \prod_{j=1}^{i-1} (z_2^j - z_1^j) + z_1$$
$$= \lambda_1 \circ Cf_{n-1}(t_2, \ldots, t_n, \lambda_2, \ldots, \lambda_n).$$

If $t_1 = 1$, we have

$$f_n = \frac{1}{2} \prod_{i=1}^{n} (z_2^i - z_1^i)x + \sum_{i=1}^{n} \frac{1}{2} z_1^i \prod_{j=1}^{i-1} (z_2^j - z_1^j) = c_1 \circ \lambda_1 \circ \ldots \circ \lambda_n.$$

If $t_k = 0$, $k > 1$, we have

$$f_n(t_1, \ldots, t_n) = \frac{1}{2} \prod_{i=1}^{n} (z_2^i - z_1^i)x + \sum_{i=1}^{n} \frac{1 + P(t_1, \ldots, t_i)}{2} z_1^i \prod_{j=1}^{i-1} (z_2^j - z_1^j)$$

$$(A) \qquad = \frac{1}{2} \prod_{i=1}^{n} (z_2^i - z_1^i)x + \sum_{i=1}^{k-2} \frac{1 + P(t_1, \ldots, t_i)}{2} z_1^i \prod_{j=1}^{i-1} (z_2^j - z_1^j)$$
$$+ \frac{1 + (1 - t_1) \ldots (1 - t_{k-1})}{2} z_1^{k-1} \prod_{j=1}^{k-2} (z_2^j - z_1^j)$$
$$+ \frac{1 + (1 - t_1) \ldots (1 - t_{k-1}) \cdot 1}{2} z_1^k \prod_{j=1}^{k-1} (z_2^j - z_1^j)$$

$$(B) \qquad + \sum_{i=k+1}^{n} z_1^i \prod_{j=1}^{i-1} (z_2^j - z_1^j)$$

$$= A + \frac{1 + (1 - t_1) \ldots (1 - t_{k-1})}{2} [z_1^k(z_2^{k-1} - z_1^{k-1}) + z_1^{k-1}]$$
$$\times \prod_{j=1}^{k-2} (z_2^j - z_1^j) + B$$
$$= f_{n-1}(t_1, \ldots, \hat{t}_k, \ldots, t_n, \lambda_1, \ldots, \lambda_{k-1} \circ \lambda_k, \ldots, \lambda_n).$$

Note that

$$\lambda_{k-1} \circ \lambda_k = (z_2^{k-1} - z_1^{k-1})(z_2^k - z_1^k)x + [z_1^{k-1} + (z_2^{k-1} - z_1^{k-1})z_1^k].$$

Finally, if $t_k = 1$, $k > 1$, we have

$$\begin{aligned}
f_n &= \frac{1}{2}\prod_{i=1}^{n}(z_2{}^i - z_1{}^i)x + \sum_{i=1}^{n}\frac{1+P(t_1,\ldots,t_n)}{2}z_1{}^i\prod_{j=1}^{i-1}(z_2{}^j - z_1{}^j) \\
&= \frac{1}{2}\prod_{i=1}^{n}(z_2{}^i - z_1{}^i)x + \sum_{i=1}^{k-1}\frac{1+P(t_1,\ldots,t_n)}{2}z_1{}^i\prod_{j=1}^{i-1}(z_2{}^j - z_1{}^j) \\
&\quad + \frac{1}{2}\sum_{i=k}^{n} z_1{}^i\prod_{j=1}^{i-1}(z_2{}^j - z_1{}^j) \\
&= f_{k-1}(t_1,\ldots,t_{k-1},\lambda_1,\ldots,\lambda_{k-1})\circ(\lambda_k\circ\ldots\circ\lambda_n).
\end{aligned}$$

The verification of the fact that the g_n also obey these relations follows from a similar computation.

One more point remains to be clarified. In this proof we used a specific point $c = \langle c_1, c_2\rangle \in \mathscr{C}(2)$. The theorem will in fact be true for any other choice of c. To see this, recall that $\mathscr{C}(2)$ is contractible; thus for c and c' in $\mathscr{C}(2)$ the map $\theta_2(c)$ is homotopic to the map $\theta_2(c')$ via any path between c and c' in $\mathscr{C}(2)$. In [**2**, Chap. V., Thm. 6.2 (ii)], it is shown that any map homotopic to an s.h.C-map is itself an s.h.C-map.

References

1. J. M. Boardman and R. M. Vogt, *Homotopy-everything H-spaces*, Bull. Amer. Math. Soc. *74* (1968), 1117–1122.

2. F. Cohen, T. Lada, and J. P. May, *The homology of iterated loop spaces*, Lecture Notes in Mathematics *533* (Springer-Verlag, 1976).

3. J. P. May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics *271* (Springer-Verlag, 1972).

4. ——— *E_∞ spaces, group completions, and permutative categories*, London Mathematical Society Lecture Note Series II, p. 61–94 (Cambridge University Press, 1974).

North Carolina State University,
Raleigh, North Carolina