

general setup :

$$c \xrightarrow{F} \mathcal{D} \quad \text{functor}$$

$$\text{Map}^h(X, Y) \xrightarrow[\cong]{?} \text{Map}^h(FX, FY)$$

(e.g., Dwyer-Kan function complexes)

Q. How much of Y can we recover from FY + extra structure?

one example of extra structure:

FY has $\text{Nat}(F, F)$ - action.

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Example (Mandell) : E_∞ -algebra structure on cochains

$$\text{Q. } \text{Map}^h(X, Y) \xrightarrow[\cong]{?} \text{Map}^h(C^*Y, C^*X)_{E_\infty \mathbb{F}_p\text{-Alg}}$$

$$C^* := C^*(-; \mathbb{F}_p)$$

\mathbb{F}_p = finite field w/ p elements.

$$\text{Map}^h(\ast, Y) \xrightarrow[\cong]{?} \text{Map}^h(C^\ast(Y), \mathbb{F}_p)$$

\mathbb{R}
 Y

$E_\infty \mathbb{F}_p\text{-Alg}$

might hope to recover Y_p^\wedge

↑

(p-completion
of Y)

Theorem (Mandell)

$$Y \cong \text{Map}^h(C^\ast(Y; \overline{\mathbb{F}}_p), \overline{\mathbb{F}}_p)$$

$E_\infty \overline{\mathbb{F}}_p\text{-Alg}$

↑ algebraic closure

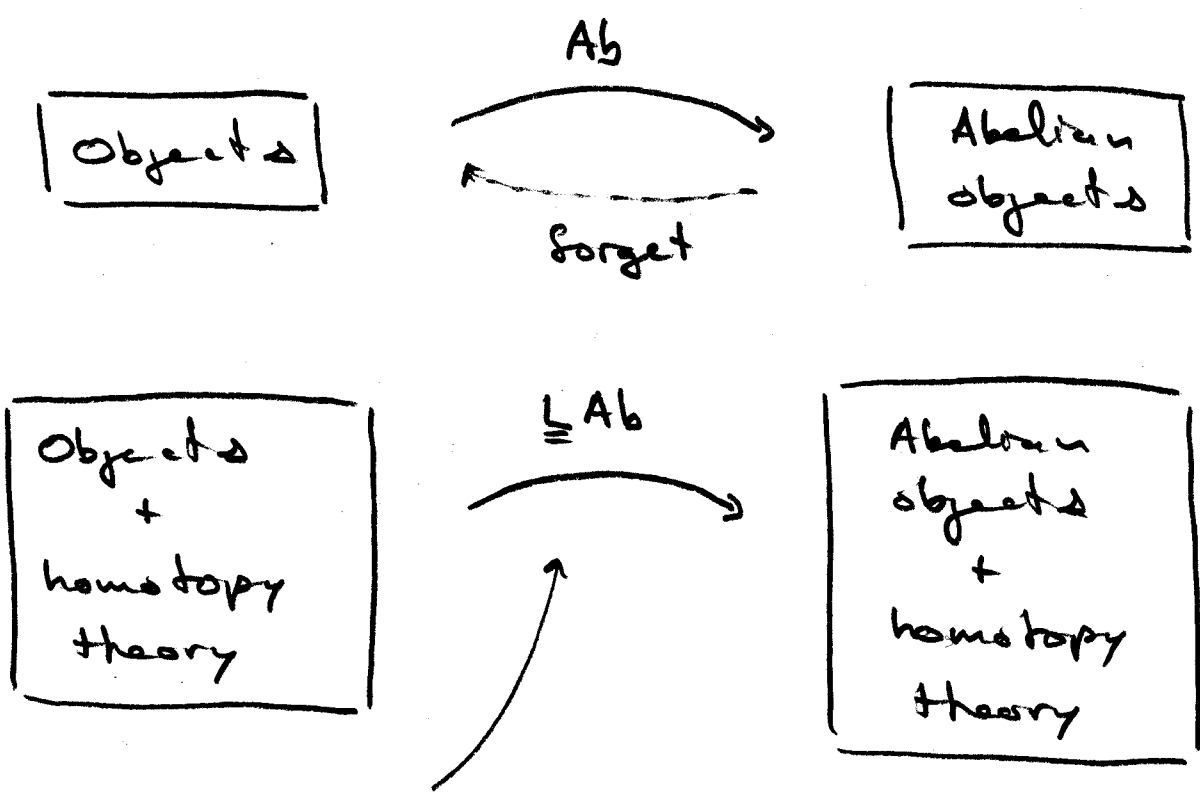
modulo

- ⎧ finiteness
- +
- nilpotency
- +
- p-completion
- +
- ⎩ connected

↑ conditions on spaces.

Q. Can we obtain analogous results for certain (derived) homology functors in other contexts?

Quillen's derived functor notion of homology



(left) derived abelianization
||
Quillen (derived) homology

Examples :

① Spaces $\Pi_n \underline{L} Ab(X) \cong H_n(X)$
 ||
 sSet $\ni X$ (singular homology of spaces)

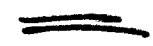
② Group $\pi_n \cong Ab(G) \cong H_{n+1}(BG)$
 ψ
 G (homology of groups)

③ Com k -algebra $\downarrow k$
 \Downarrow equiv. $\pi_n \cong Ab(X) =$ (Andre' - Quillen homology)

(Non-unital com k -algebra) $=:$ NUCA

ψ
 X

This algebraic structure is parametrized by an operad.



(Q. what does abelianization look like?)

$$\begin{array}{ccc} \text{NUCA} & \begin{array}{c} \xrightarrow{Ab} \\ \xleftarrow{\text{forget}} \end{array} & (\text{NUCA})_{ab} = k\text{-Mod} \\ \psi & & \\ X & & \end{array}$$

$$Ab(X) = X/X^2$$

Example (3) generalizes to :

Algebras (\neq modules) over operads

Two contexts :

(Ch_k, \otimes, k)

(Sp^{Σ}, \wedge, S)

unbounded chain complexes

symmetric spectra

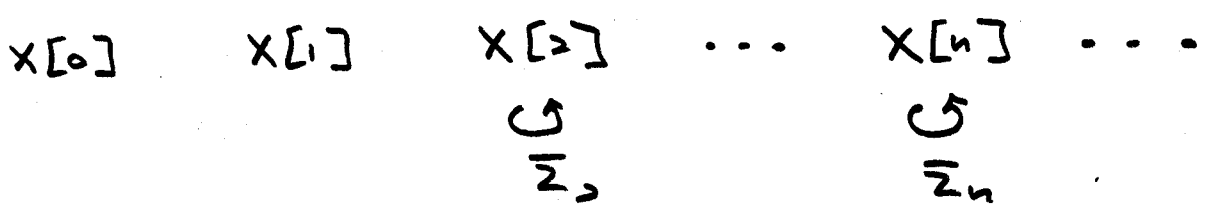
$k = \text{field of char. } 0$

(Q. What does $\underline{\mathbb{L}} Ab$ look like?)

recall :

symmetric sequence
of chain complexes

X is a sequence
(resp. symmetric spectra)



equipped w/ Σ -action.

Monoidal structure --- on
sym. sequences

$X \in \text{SymSeq}$ $A \in \text{Ch}_k$ (resp. $\text{Sp}^{\mathbb{Z}}$)

define $X(A) := \coprod_{t \geq 0} X[t] \otimes_{\mathbb{Z}} A^{\otimes t}$ in Ch_k

For $X, Y \in \text{SymSeq}$ define $X \circ Y$
in SymSeq s.t.

$$X(Y(A)) \cong \underbrace{(X \circ Y)(A)}_{\uparrow} \quad \forall A$$

(looks like composition
of power series)

$$I = (0, k, 0, 0, 0, \dots)$$

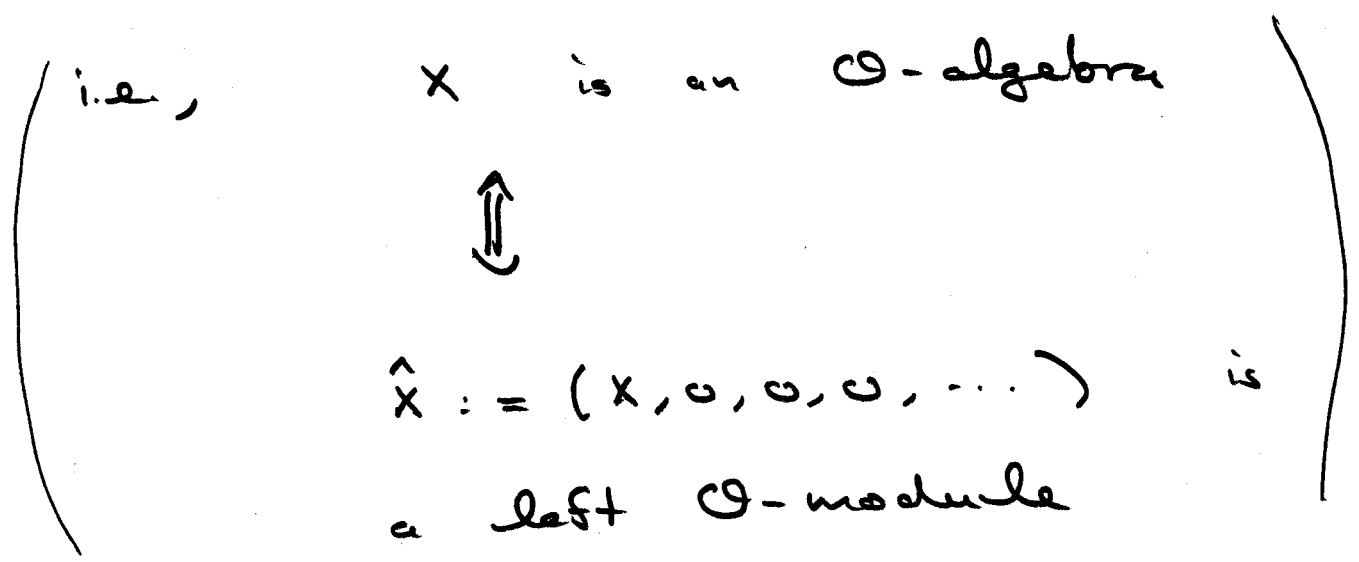
$$X \circ I \cong X \cong I \circ X$$

(two-sided unit)

- Operad \mathcal{O} is a monoid in Sym Seq w.r.t. --- .
- left \mathcal{O} -module is a $\text{sym. seq. } X$ equipped w/ a left \mathcal{O} -action

$$(\mathcal{O} \times X \xrightarrow{m} X)$$

• \mathcal{O} -algebra (as defined in Kriegl-May) is the same as a left \mathcal{O} -module concentrated at \mathcal{O}



\equiv

$\left(\text{Q. what does } \underline{\text{Ab}}(X) \text{ look like for } \mathcal{O}\text{-algebras } X? \right)$

$\mathcal{O} \xrightarrow{F} I$ augmented operad \mathcal{O}
in Ch_k .

$$Alg_{\mathcal{O}} \begin{array}{c} \xrightarrow{Ab} \\ \xleftarrow{\text{Forget}} \\ \parallel \\ f^* \end{array} (Alg_{\mathcal{O}})_{ab} = Ch_k = Alg_I$$

provided that
 $\mathcal{O}[0] = 0$
 $\mathcal{O}[1] = k$

$\therefore Ab = f^* = I \begin{array}{c} 0 \\ \mathcal{O} \end{array} -$ "indecomposables"

$\underline{\underline{Ab}} = \underline{\underline{f^*}} = I \begin{array}{c} 0 \\ \mathcal{O} \end{array} -$ derived "indecomposables"

\parallel
Quillen's (derived)
homology

Main Question (Analogous to Mandell's thm.)

① How much of X can be recovered
from its derived homology
 $I \begin{array}{c} 0 \\ \mathcal{O} \end{array} X +$ extra structure?

② What is the extra structure?

There are several things one has to do :

- ① Establish a homotopy theory on \mathcal{O} -algebras s.t. Quillen homology $I_{\mathcal{O}}^L$ is well-defined.
- ② calculate $I_{\mathcal{O}}^L X$ as a simplicial bar construction.
- ③ candidate construction for recovering X from $I_{\mathcal{O}}^L X$ + extra structure.

↖ Approach: find conditions s.t.

$$X \xrightarrow[\sim]{?} \text{holim}_{\Delta} \left(I_{\mathcal{O}}^L X \rightrightarrows I_{\mathcal{O}}^L (I_{\mathcal{O}}^L X) \rightrightarrows \dots \right)$$

partial motivation
 (from Koszul duality)

idea is : cosimplicial diagram encodes extra structure on $I_{\mathcal{O}}^L X$

Motivation \neq results

Sor ① \neq ②

Theorem 1 (H.)

If \mathcal{O} is an operad in symmetric spectra (or Ch_k) then

(a) \mathcal{O} -algebras (resp. $Lt_{\mathcal{O}}$) has a naturally occurring model cat. structure

$$(b) \quad Ho(Alg_{\mathcal{O}}) \xrightleftharpoons[\mathcal{O}]{I_{\mathcal{O}}^L} Ho(\text{Symmetric spectra})$$

Quillen homology is well-defined

(P. How to calculate Quillen homology?)

By definition,

$$I_{\mathcal{O}} X := \text{colim} \left(\underbrace{I \circ X}_{\cong X} \begin{matrix} \xrightarrow{\text{mod}} \\ \xleftarrow{\text{mod}} \end{matrix} \underbrace{I \circ \mathcal{O} \circ X}_{\cong \mathcal{O} \circ X} \right)$$

$$\text{colim}_{\Delta^op} \left(I \circ X \rightleftharpoons I \circ \mathcal{O} \circ X \rightleftharpoons I \circ \mathcal{O} \circ \mathcal{O} \circ X \rightleftharpoons \dots \right)$$

\parallel
 $Bar^{\circ}(I, \mathcal{O}, X) = \left(\begin{matrix} \text{simplicial bar} \\ \text{construction} \end{matrix} \right)$

$$\left(\begin{array}{l} \text{If } H_0 \text{ is true,} \\ H_0 \text{ for } X \stackrel{?}{\cong} \text{hocolim}_{\Delta^{op}} \text{Bar}^{\circ}(I, \mathcal{O}, X) \end{array} \right)$$

$\cong ?$

$|\text{Bar}^{\circ}(I, \mathcal{O}, X)|$

rule: when X is concentrated at 1, this reduces to calculation of

$$\text{Tor}_{\mathbb{O}[1]}^*(k, X[1])$$

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Theorem 2 (H.)

Let \mathcal{O} be an operad in symmetric spectra (or Chk).

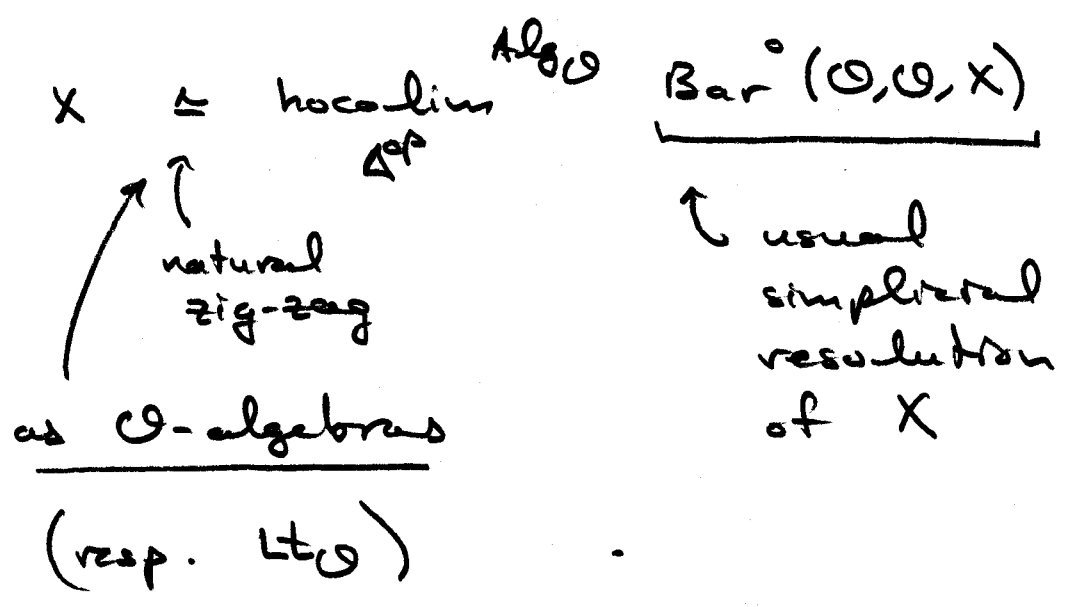
(a) If B is a simplicial \mathcal{O} -algebra (or sLtg) then

$$\mathbb{F} \text{ hocolim}_{\Delta^{op}}^{\text{Algs}} B \cong \text{hocolim}_{\Delta^{op}} \mathbb{F} B$$

nat'l
zig-zag

\mathbb{F} = forgetful functor

(b) If X is an \mathcal{O} -algebra (or $Lt_{\mathcal{O}}$)
 then



(c) If X is \mathcal{O} -algebra (resp. $Lt_{\mathcal{O}}$)

(i) For $Ch_k = I_{\mathcal{O}}^L X \simeq | \text{Bar}^{\circ}(I, \mathcal{O}, X) |$

(ii) For symmetric spectra: true if
 extra cofibrancy conditions hold

(e.g., $\mathcal{O} = \text{cofibrant operad}$
 $X = \text{cofibrant } \mathcal{O}\text{-algebra}$)

mk: Results hold more generally:
 $R \xrightarrow{F} S$ morphism of operads

$$\text{Alg}_R \xrightleftharpoons[f^*]{f_*} \text{Alg}_S \quad \underline{f}_*(X) \simeq | \text{Bar}^{\circ}(S, R, X) |$$

Proof of Theorem 2 : (Key steps)

Reduction to gluing on cells in
simpler $\mathcal{O}\text{-Alg}$ ($H_{\mathcal{O}}^{\text{set}}$)

$$\begin{array}{ccc}
 \mathcal{O} \circ \tilde{X} & \longrightarrow & A \\
 \downarrow & \text{push} & \downarrow \\
 \mathcal{O} \circ \tilde{Y} & \longrightarrow & A \amalg_{\mathcal{O} \circ \tilde{X}} \mathcal{O} \circ \tilde{Z}
 \end{array}
 \quad (*) \quad \text{in } (\text{Alg}_{\mathcal{O}})^{\text{set}}$$

\parallel claim : in underlying category

$$\text{colim} \left(\underbrace{A_0}_{\parallel A} \rightarrow A_1 \rightarrow A_2 \rightarrow \dots \right)$$

$A_{-1} := *$

↑ (filtration motivated by Elmendorf-Mandell)

which satisfies two properties :

① Associated graded

$$\amalg_{t \in \mathbb{Z}} A_t / A_{t-1} \cong A \amalg_{\mathcal{O} \circ \tilde{Z}} \mathcal{O} \circ \tilde{Z}$$

for some simpler object \tilde{Z}

② IF we apply $\text{colim}_{\Delta^{\text{op}}} (-) =: \pi_0(-)$ to (*)

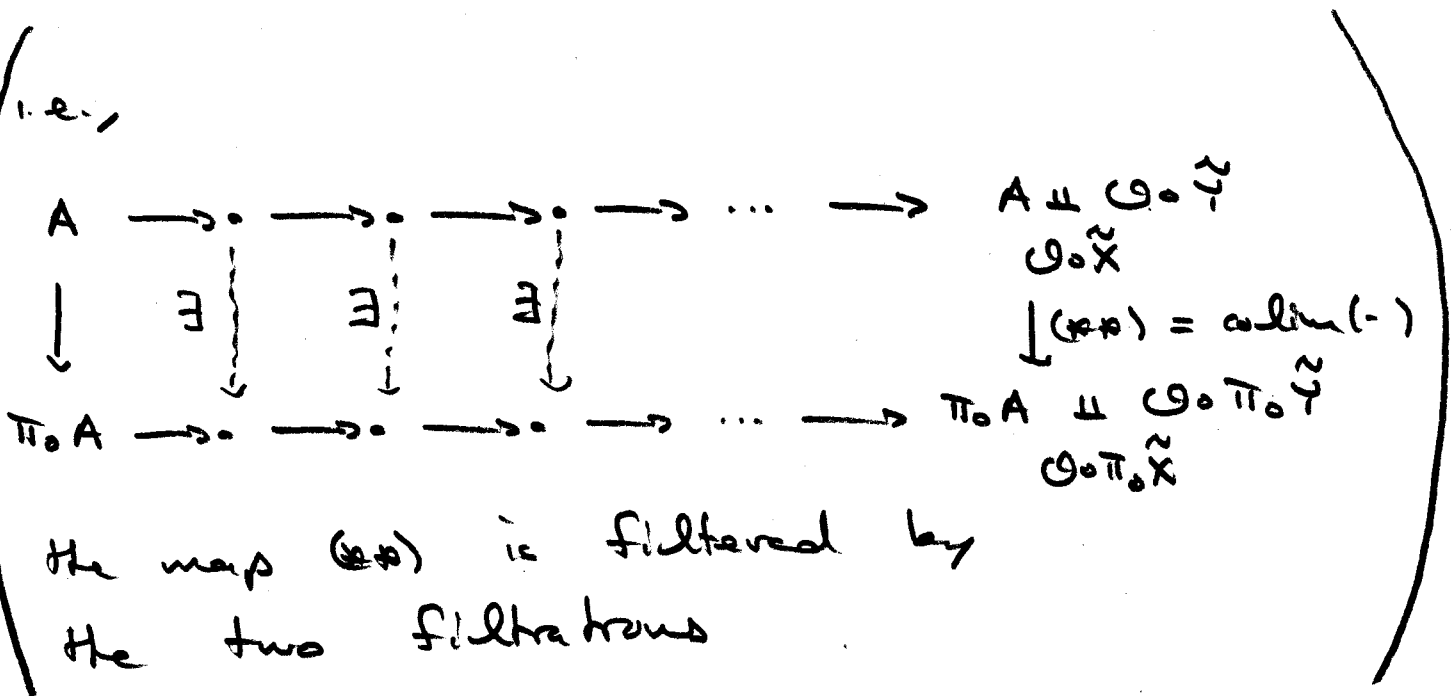
$$\begin{array}{ccc}
 \mathcal{O} \circ \pi_0 \tilde{X} & \longrightarrow & \pi_0 A \\
 \downarrow & \text{push} & \downarrow \\
 \mathcal{O} \circ \pi_0 \tilde{Y} & \longrightarrow & \pi_0 A \amalg_{\mathcal{O} \circ \pi_0 \tilde{X}} \mathcal{O} \circ \pi_0 \tilde{Y}
 \end{array}$$

||? claim: in underlying category

$$\text{colim} \left(\underbrace{(\pi_0 A)_0}_{\cong \pi_0 A} \rightarrow (\pi_0 A)_1 \rightarrow (\pi_0 A)_2 \rightarrow \dots \right)$$

then the natural map

$$A \longrightarrow \pi_0 A \quad \text{respects the filtrations}$$



Q. How does this get used?

Want to prove: $\mathbb{F} \text{ hocolim}_{\Delta^{\text{op}}} \text{Algo } B \stackrel{?}{\simeq} |\mathbb{F}B|$

For every simplicial \mathcal{O} -algebra B

Reduces to checking:

$\mathbb{F} \text{ colim}_{\Delta^{\text{op}}} \text{Algo } Z \stackrel{?}{\simeq} |\mathbb{F}Z|$

For every cofibrant diagram Z

↑ built by attaching cells

By appropriate induction argument, the filtration reduces to checking:

$|A \amalg \mathcal{O} \cdot \tilde{Z}|$

$\simeq \downarrow ?$

$\pi_0 A \amalg \mathcal{O} \cdot \pi_0 \tilde{Z}$

(induction pushes things off to ∞ , \neq reduces to starting induction
w/ $A = \mathcal{O} \cdot \Delta[\sigma_0]$
 $\neq \tilde{Z} = Y/X \cdot \Delta[n]$)

Observation : $\left(\begin{array}{l} \text{coproduct } \perp \text{ of} \\ \text{simplicial homotopies} \end{array} \right)$

||

$\left(\text{a simplicial homotopy} \right)$

↖ This finishes the proof.

Remark :

- Property ① of filtration works for any shape diagram.
- Property ② depends on shape of Δ^{op}

↖ $\left(\begin{array}{l} \text{e.g., } \textcircled{2} \text{ fails to be true} \\ \text{for coproducts} \end{array} \right)$

$\left(\begin{array}{l} \text{e.g., } \Phi(A \amalg B) \not\cong \Phi A \amalg \Phi B \\ \uparrow \text{ in general.} \end{array} \right)$