general setup :

$\mathcal{C} \xrightarrow{F} \mathcal{D}$ functor

$$\mathrm{Map}^h(X,Y) \underset{\simeq}{\overset{?}{\longrightarrow}} \mathrm{Map}^h(FX,FY)$$

(e.g., Dwyer-Kan function complexes.)

(Q. How much of Y can we recover from FY + extra structure ?)

one example of extra structure :

FY has $\mathrm{Nat}(F,F)$ - action .

Example (Mandell) : E_∞ - algebra structure on cochains

Q. $\mathrm{Map}^h(X,Y) \underset{\simeq}{\overset{?}{\longrightarrow}} \mathrm{Map}^h_{E_\infty \mathbb{F}_p\text{-Alg}}(C^*Y, C^*X)$

$C^* := C^*(-;\mathbb{F}_p)$ $\qquad$ $\mathbb{F}_p$ = finite field w/ p elements.

$$\begin{array}{ccc} \mathrm{Map}^h(*, Y) & \xrightarrow[\simeq]{?} & \underset{E_\infty\, \mathbb{F}_p\text{-Alg}}{\mathrm{Map}^h}(C^* Y, \mathbb{F}_p) \\ \wr\wr & & \\ Y & & \end{array}$$

might hope to recover $\hat{Y_p}$

(p-completion of Y)

<u>Theorem</u> (Mandell)

$$Y \simeq \underset{E_\infty\, \overline{\mathbb{F}}_p\text{-Alg}}{\mathrm{Map}^h}(C^*(Y; \overline{\mathbb{F}}_p), \underbrace{\overline{\mathbb{F}}_p})$$

algebraic closure

modulo

$$\left\{ \begin{array}{l} \text{finiteness} \\ \quad + \\ \text{nilpotency} \\ \quad + \\ \text{p-completion} \\ \quad + \\ \text{connected} \end{array} \right.$$

conditions on spaces.

(Q. Can we obtain analogous results for certain (derived) homology functors in other contexts?)

Quillen's derived functor notion of homology

$$\begin{tikzcd} \boxed{\text{Objects}} \arrow[r, bend left, "Ab"] & \boxed{\begin{array}{c}\text{Abelian}\\ \text{objects}\end{array}} \arrow[l, bend left, "\text{forget}"] \end{tikzcd}$$

$$\begin{tikzcd} \boxed{\begin{array}{c}\text{Objects}\\ +\\ \text{homotopy}\\ \text{theory}\end{array}} \arrow[r, bend left, "\underline{\underline{L}}Ab"] & \boxed{\begin{array}{c}\text{Abelian}\\ \text{objects}\\ +\\ \text{homotopy}\\ \text{theory}\end{array}} \end{tikzcd}$$

(left) derived abelianization

$\shortparallel$

Quillen (derived) homology

Examples:

① Spaces $\qquad \pi_n \, \underline{\underline{L}}Ab(X) \cong H_n(X)$

$\shortparallel$

sSet $\ni X$ $\qquad$ (singular homology of spaces)

(2) Group $\quad \pi_n \mathbb{L} Ab(G) \cong H_{n+1}(BG)$

$\in$

G

(homology of groups)

(3) Com k-algebra $\downarrow k$ $\quad \pi_n \mathbb{L} Ab(X) = $ (André–Quillen homology)

$\Updownarrow$ equiv.

$\begin{pmatrix} \text{Non-unital} \\ \text{com } k\text{-algebra} \end{pmatrix} =: NUCA$

$\in$

X

This algebraic structure is parametrized by an operad.

(Q. What does abelianization look like?)

$$\begin{array}{c} NUCA \\ \in \\ X \end{array} \underset{\text{forget}}{\overset{Ab}{\rightleftarrows}} (NUCA)_{ab} = k\text{-Mod}$$

$$Ab(X) = X/X^2$$

Example 3 generalizes to:

<u>Algebras ($\neq$ modules)</u> over operads

<u>Two contexts</u>:

$(Ch_k, \otimes, k)$

unbounded chain complexes

k = field of char. 0

$(Sp^{\Sigma}, \wedge, S)$

symmetric spectra

($\underline{Q}$. What does $\underline{\underline{L}}Ab$ look like?)

<u>recall</u>:

symmetric <u>sequence</u> X is a sequence of chain complexes (resp. symmetric spectra)

$$\begin{array}{cccccc} X[0] & X[1] & X[2] & \cdots & X[n] & \cdots \\ & & \circlearrowleft & & \circlearrowleft & \\ & & \Sigma_2 & & \Sigma_n & \end{array}$$

equipped w/ Σ-action.

Monoidal structure $- \circ -$ on sym. sequences

$X \in \mathrm{SymSeq}$ $\qquad A \in \mathrm{Ch}_k$ $\qquad$ (resp. Sp^{Σ})

define $\quad X(A) := \coprod_{t \geq 0} X[t] \otimes_{\Sigma_t} A^{\otimes t}$ $\quad$ in Ch_k

For $X, Y \in \mathrm{SymSeq}$ define $X \circ Y$

in SymSeq s.t.

$$X(Y(A)) \cong \underbrace{(X \circ Y)}(A) \quad \forall\ A$$

(looks like composition of power series)

$$I = (0, k, 0, 0, 0, \ldots)$$

$$X \circ I \cong X \cong I \circ X$$

(two-sided unit).

- <u>Operad</u> $\mathcal{O}$ is a monoid in SymSeq w.r.t. $- \circ -$.

- <u>left $\mathcal{O}$-module</u> is a sym. seq. X equipped w/ a left $\mathcal{O}$-action

$$\left(\mathcal{O} \circ X \xrightarrow{m} X \right)$$

- <u>$\mathcal{O}$-algebra</u> (as defined in Kriz-May) is the same as a left $\mathcal{O}$-module concentrated at 0

$$\left(\begin{array}{l} \text{i.e., } \quad X \text{ is an } \mathcal{O}\text{-algebra} \\ \qquad\qquad \Updownarrow \\ \qquad \hat{X} := (X, \emptyset, \emptyset, \emptyset, \ldots) \text{ is} \\ \qquad \text{a left } \mathcal{O}\text{-module} \end{array} \right)$$

(<u>Q</u>. what does $\underline{\underline{L}}Ab(X)$ look like for $\mathcal{O}$-algebras X?)

$\mathcal{O} \xrightarrow{f} I$ augmented operad $\mathcal{O}$ in Ch_k.

$$\mathrm{Alg}_{\mathcal{O}} \underset{\substack{\text{forget} \\ \| \\ f^*}}{\overset{\mathrm{Ab}}{\rightleftarrows}} (\mathrm{Alg}_{\mathcal{O}})_{ab} = \mathrm{Ch}_k = \mathrm{Alg}_I$$

provided that
$\mathcal{O}[0] = 0$
$\mathcal{O}[1] = k$

$$\therefore \mathrm{Ab} = f_* = I \circ_{\mathcal{O}} - \qquad \text{"indecomposables"}$$

$$\mathbb{L}\mathrm{Ab} = \mathbb{L} f_* = I \circ^{\mathsf{L}}_{\mathcal{O}} - \qquad \text{derived "indecomposables"}$$

$\|$

Quillen's (derived) homology

<u>Main Question</u> (Analogous to Mandell's thm.)

(1) How much of X can be recovered from its derived homology $I \circ^{\mathsf{L}}_{\mathcal{O}} X$ + extra structure?

(2) What is the extra structure?

There are several things one has to do:

(1) Establish a homotopy theory on $\mathcal{O}$-algebras s.t. Quillen homology $I \circ^{L}_{\mathcal{O}} -$ is well-defined.

(2) Calculate $I \circ^{L}_{\mathcal{O}} X$ as a simplicial bar construction.

(3) Candidate construction for recovering X from $I \circ^{L}_{\mathcal{O}} X$ + extra structure.

Approach: find conditions s.t.

$$X \underset{\sim}{\overset{?}{\longrightarrow}} \operatorname{holim}_{\Delta} \left(I \circ^{L}_{\mathcal{O}} X \rightrightarrows I \circ^{L}_{\mathcal{O}} (I \circ^{L}_{\mathcal{O}} X) \Rrightarrow \cdots \right)$$

(partial motivation from Koszul duality)

(<u>idea is</u>: cosimplicial diagram encodes extra structure on $I \circ^{L}_{\mathcal{O}} X$)

Motivation & results for ① & ②

Theorem 1 (H.)

If $\mathcal{O}$ is an operad in symmetric spectra (or Ch_k) then

(a) $\mathcal{O}$-algebras (resp. $Lt_{\mathcal{O}}$) has a naturally occuring model cat. structure

(b) $$\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}) \underset{}{\overset{I \circ^{L}_{\mathcal{O}} -}{\rightleftarrows}} \mathrm{Ho}(\text{Symmetric spectra})$$

Quillen homology is well-defined

(Q. How to calculate Quillen homology?)

By definition,

$$I \circ_{\mathcal{O}} X := \operatorname{colim}\Big(\underbrace{I \circ X}_{\cong\, X} \underset{\text{idom}}{\overset{\text{moid}}{\leftleftarrows}} \underbrace{I \circ \mathcal{O} \circ X}_{\cong\, \mathcal{O} \circ X} \Big)$$

$$\simeq$$

$$\operatorname*{colim}_{\Delta^{op}} \Big(\underbrace{I \circ X \leftleftarrows I \circ \mathcal{O} \circ X \Lleftarrow I \circ \mathcal{O} \circ \mathcal{O} \circ X \Lleftarrow \cdots}_{=\ \mathrm{Bar}^{\circ}(I, \mathcal{O}, X)} \Big)$$

$\mathrm{Bar}^{\circ}(I,\mathcal{O},X) = \left(\begin{array}{c}\text{simplicial bar}\\ \text{construction}\end{array}\right)$

(Q. Is it true,

$$I \circ^{L}_{\mathcal{O}} X \overset{?}{\simeq} \operatorname{hocolim}_{\Delta^{op}} \mathrm{Bar}^{\circ}(I, \mathcal{O}, X)$$)

$$\simeq \big| ?$$

$$|\mathrm{Bar}^{\circ}(I, \mathcal{O}, X)|$$

rmk: when X is concentrated at 1, this reduces to calculation of $\mathrm{Tor}^{*}_{\mathcal{O}[1]}(k, X[1])$

Theorem 2 (H.)

Let $\mathcal{O}$ be an operad in symmetric spectra (or Ch_k).

(a) If B is a simplicial $\mathcal{O}$-algebra (or $sLt_{\mathcal{O}}$) then

$$\Phi \operatorname{hocolim}^{Alg_{\mathcal{O}}}_{\Delta^{op}} B \underset{\substack{\text{nat'l} \\ \text{zig-zag}}}{\simeq} \operatorname{hocolim}_{\Delta^{op}} \Phi B$$

Φ = forgetful functor

(b) If X is an $\mathcal{O}$-algebra (or $\mathrm{Lt}_{\mathcal{O}}$) then

$$X \simeq \operatorname{hocolim}^{\mathrm{Alg}_{\mathcal{O}}}_{\Delta^{op}} \underbrace{\mathrm{Bar}^{\circ}(\mathcal{O},\mathcal{O},X)}$$

natural zig-zag

usual simplicial resolution of X

as $\mathcal{O}$-algebras

(resp. $\mathrm{Lt}_{\mathcal{O}}$).

(c) If X is $\mathcal{O}$-algebra (resp. $\mathrm{Lt}_{\mathcal{O}}$)

(i) for Ch_k: $I \circ^{L}_{\mathcal{O}} X \simeq |\mathrm{Bar}^{\circ}(I,\mathcal{O},X)|$

(ii) for symmetric spectra: true if extra cofibrancy conditions hold

(e.g., $\mathcal{O}$ = cofibrant operad
X = cofibrant $\mathcal{O}$-algebra)

rmk: Results hold more generally:

$R \xrightarrow{f} S$ morphism of operads

$$\mathrm{Alg}_R \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathrm{Alg}_S \qquad Lf_*(X) \simeq |\mathrm{Bar}^{\circ}(S,R,X)|.$$

Proof of Theorem 2 : (Key steps)

Reduction to gluing on cells in simplicial $\mathcal{O}$-Alg ($\mathcal{H}_{\mathcal{O}}^{\Delta^{op}}$)

$$\begin{tikzcd}
\mathcal{O}\circ\tilde{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{push}"] & A \arrow[d] \\
\mathcal{O}\circ\tilde{Y} \arrow[r] & A \amalg_{\mathcal{O}\circ\tilde{X}} \mathcal{O}\circ\tilde{Y}
\end{tikzcd} \quad (*) \quad \text{in } (\mathrm{Alg}_{\mathcal{O}})^{\Delta^{op}}$$

$\cong$ claim : in underlying category

$$\mathrm{colim}\big(\underbrace{A_0}_{\cong A} \to A_1 \to A_2 \to \cdots \big)$$

(filtration motivated by Elmendorf - Mandell)

$A_{-1} := *$

which satisfies two properties :

(1) Associated graded

$$\coprod_t A_t / A_{t-1} \cong A \amalg \mathcal{O}\circ\tilde{Z}$$

for some simplicial object $\tilde{Z}$

② If we apply $\operatorname{colim}_{\Delta^{op}}(-) =: \pi_0(-)$

to (✱)

$$
\begin{tikzcd}
\mathcal{O}_0 \pi_0 \tilde{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{push}"] & \pi_0 A \arrow[d] \\
\mathcal{O}_0 \pi_0 \tilde{Y} \arrow[r] & \pi_0 A \amalg_{\mathcal{O}_0 \pi_0 \tilde{X}} \mathcal{O}_0 \pi_0 \tilde{Y}
\end{tikzcd}
$$

$\cong$ claim: in underlying category

$$\operatorname{colim}\Big(\underbrace{(\pi_0 A)_0}_{\substack{\cong \\ \pi_0 A}} \to (\pi_0 A)_1 \to (\pi_0 A)_2 \to \cdots\Big)$$

then the natural map

$A \to \pi_0 A$ respects the filtrations

(i.e.,

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d, dashed, "\exists"'] & \bullet \arrow[r] \arrow[d, dashed, "\exists"'] & \bullet \arrow[r] \arrow[d, dashed, "\exists"'] & \cdots \arrow[r] & A \amalg_{\mathcal{O}_0 \tilde{X}} \mathcal{O}_0 \tilde{Y} \arrow[d, "(✱✱) = \operatorname{colim}(-)"] \\
\pi_0 A \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r] & \pi_0 A \amalg_{\mathcal{O}_0 \pi_0 \tilde{X}} \mathcal{O}_0 \pi_0 \tilde{Y}
\end{tikzcd}
$$

the map (✱✱) is filtered by the two filtrations)

(Q. How does this get used?)

Want to prove: $\Phi\, \mathrm{hocolim}^{\mathrm{Alg}_{\mathcal{O}}}_{\Delta^{op}} B \overset{?}{\simeq} |\Phi B|$

for every simplicial $\mathcal{O}$-algebra B

<u>Reduces to checking</u>:

$\Phi\, \mathrm{colim}^{\mathrm{Alg}_{\mathcal{O}}}_{\Delta^{op}} Z \overset{?}{\simeq} |\Phi Z|$

for every <u>cofibrant diagram Z</u>

↖ built by attaching cells

By appropriate induction argument, the filtration reduces to checking:

$$\begin{array}{c} |A \amalg \mathcal{O}\circ \tilde{Z}| \\ \simeq \big\downarrow ? \\ \pi_0 A \amalg \mathcal{O}\circ \pi_0 \tilde{Z} \end{array}$$

(induction pushes things off to ∞, & reduces to starting induction w/ $A = \mathcal{O}\circ *\cdot \Delta[0]$ & $\tilde{Z} = Y/X \cdot \Delta[n]$)

<u>Observation</u> :

$$\left(\begin{array}{l}\text{coproduct } \amalg \text{ of} \\ \text{simplicial homotopies}\end{array}\right)$$

$$\|$$

$$\left(\text{a simplicial homotopy}\right)$$

this finishes the proof.

<u>Remark</u> :

- Property ① of filtration works for any shape diagram.

- Property ② depends on shape of Δ^{op}

(e.g., ② <u>fails to be true</u> for coproducts)

$$\left(\text{e.g., } \quad \Phi(A \overset{h}{\amalg} B) \not\simeq \Phi A \overset{h}{\amalg} \Phi B \quad \text{in general.}\right)$$