On the Transfer of Multiplicative Structure

Joint work with Samson Saneblidze

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Special Session on Homotopical Algebra
with Applications to Mathematical Physics

April 4, 2008

A–infinity Maps and Multiplihedra

- Given A_∞-algebras A and B, a morphism

$$F = f_1 + f_2 + f_3 + \cdots : B \Rightarrow A$$

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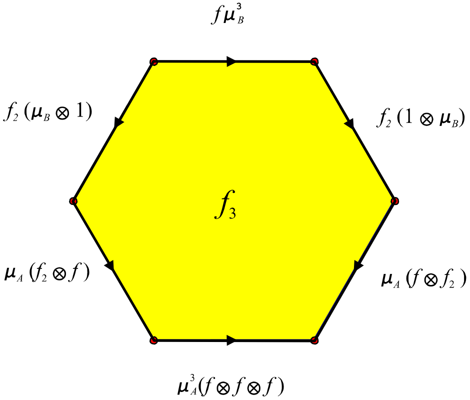
- Components of the coboundary

$$\nabla f_2 = \mu_A (f \otimes f) - f \mu_B$$

  are identified with the endpoints of J_2

Multiplihedra

- J_3 is a hexagonal plane region



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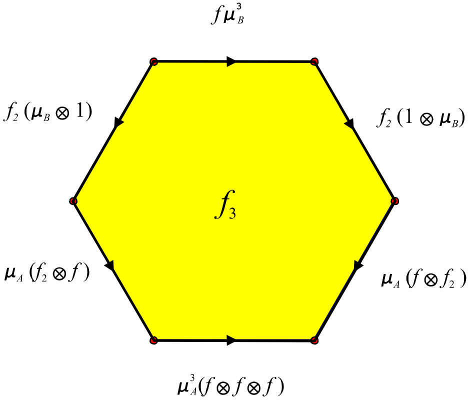

- $\nabla f_3 = \mu_A^3 f^{\otimes 3} + \mu_A (f_2 \otimes f - f \otimes f_2) + f_2 (\mu_B \otimes 1 - 1 \otimes \mu_B) - f \mu_B^3$

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Multiplihedra

- J_n is an $(n-1)$-dimensional polyhedron
- We are especially interested in the cochain

$$\ominus_n = \nabla f_n + f \mu_B^n \in Hom^{n-2}\left(B^{\otimes n}, A\right)$$

Main Theorem

Let A and B be DGMs over a commutative ring with unity

- A chain map $f : B \to A$ induces a cochain map

$$\bar{f} : \left(Hom\left(B^{\otimes n}, B\right); \nabla_B\right) \to \left(Hom\left(B^{\otimes n}, A\right); \nabla\right)$$

  given by $\bar{f}(u) = fu$

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- **Theorem 1**. *Let A be an A_∞-algebra and let $f : B \to A$* be a chain map. *If $\bar{f}$ is a quasi-isomorphism, then*

  (i) f transfers the A_∞-algebra structure from A to B; the induced structure on B is unique up to automorphism

  (ii) There is a map $F : B \Rightarrow A$ of A_∞-algebras *extending f*

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- Below A and B have *torsion* and f has *no* right-homotopy inverse

Example 1

$$
\begin{array}{ccccccccccccc}
M^0 & \longrightarrow & 0 & \longrightarrow & M^2 & \longrightarrow & M^3 & \longrightarrow & M^4 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\mathbb{Z} & & & & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & & \mathbb{Z}_4 & & \mathbb{Z}_2 & & & & \\
 & & & & (a_2, b_2) & \longmapsto & (0, 2a_3) & & & & & & \\
 & & & & & & a_3 & \longmapsto & a_4 & & & &
\end{array}
$$

- $A = T^a M \big/ \left(a_2^2 + a_4,\ a_4 a_3 + a_3 a_4,\ (a_2 a_3 + a_3 a_2)^2\right)$

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- $A = T^a M \,/\, \left(a_2^2 + a_4,\ a_4 a_3 + a_3 a_4,\ (a_2 a_3 + a_3 a_2)^2\right)$
- A is an A_∞-algebra with trivial higher order structure

The Cup Product

- $H^n(A) = \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}_2 & n = 2, 5, 7 \\ 0 & \text{otherwise} \end{cases}$

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- $\mu_H(u \otimes v) = \mu_H(v \otimes u) = w$

The Transfer Map

- Define $f : H(A) \to A$ by

$$\begin{aligned} f(u) &= a_2 \\ f(v) &= a_2 a_3 + a_3 a_2 \\ f(w) &= a_2(a_2 a_3 + a_3 a_2) \end{aligned}$$

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- f has no right-homotopy inverse
- $\bar{f}$ is a quasi-isomorphism

Transfer of Structure

- Define $F = f_1 + f_2 + \cdots$ by

$$
\begin{aligned}
f_1 &= f \\
f_2\left(u, u\right) &= a_3 \\
f_n &= 0, \ \ n \geq 3
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- Then $\mu_H^3(u \otimes u \otimes u) = v$
- Define $\mu_H^n = 0$ for $n \geq 4$
- (H, μ_H, μ_H^3) is an A_∞-algebra and F is an A_∞-map

Proof of Theorem 1: Transferring DGA Structure

Given $(A, d, \mu, \mu^n)_{n\geq 3}$ and $f : B \to A$ such that $\bar{f}$ is a quasi-isomorphism

- Let $\ominus_2 = \mu(f \otimes f)$; then $\nabla\mu = 0$ implies $\nabla\ominus_2 = 0$

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- Furthermore $[\ominus_2] - \bar{f}_*[\mu_B] = [\ominus_2 - f\mu_B] = [0]$
- Choose f_2 such that $\nabla f_2 = \ominus_2 - f\mu_B = \mu\,(f \otimes f) - f\mu_B$

Proof of Theorem 1: Transferring DGA Structure

$$\begin{array}{ccc} \mu_B(d_B \otimes 1 + 1 \otimes d_B) - d_B \mu_B & \longleftarrow & \mu_B \in (\bar{f}_*)^{-1}[\ominus_2] \\ \| & & \downarrow \bar{f} \\ 0 & \longleftarrow & \left\{ \begin{array}{c} f\mu_B = \ominus_2 - \nabla f_2 \\ \ominus_2 \end{array} \right. \end{array}$$

$$\nabla f_2 = \ominus_2 - f\mu_B$$

- Then (B, d_B, μ_B) is a DGA and $F = f + f_2$ is an A_2-map

Proof of Theorem 1: Transferring Higher Order Structure

- Let $\ominus_3 = \mu^3 f^{\otimes 3} + \mu \left(f_2 \otimes f - f \otimes f_2 \right) + f_2 \left(\mu_B \otimes 1 - 1 \otimes \mu_B \right)$

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- Choose w such that $\nabla_B(w) = \mu_B\left(1 \otimes \mu_B - \mu_B \otimes 1\right)$
- Then $\nabla \ominus_3 = f\nabla_B(w) = \left(\bar{f}\nabla_B\right)(w) = \left(\nabla \bar{f}\right)(w) = \nabla\left(fw\right)$

Proof of Theorem 1 (continued)

- Consider the cocycle $\ominus_3 - fw \in Hom^2\left(B^{\otimes 3}, B\right)$

Proof of Theorem 1 (continued)

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Proof of Theorem 1 (continued)

- Consider the cocycle $\ominus_3 - fw \in Hom^2\left(B^{\otimes 3}, B\right)$
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- Define $\mu_B^3 = w + z$

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- Define $\mu_B^3 = w + z$
- Then $\left[\ominus_3 - f\mu_B^3\right] = \left[\ominus_3 - f\left(w + z\right)\right]$

$$= \left[\ominus_3 - f\left(w\right) - \left(\ominus_3 - fw + \nabla u\right)\right] = \left[0\right]$$

Proof of Theorem 1 (continued)

- Consider the cycle $\ominus_3 - fw \in Hom^2\left(B^{\otimes 3}, B\right)$
- Choose $z \in \left(\bar{f}_*\right)^{-1}\left[\ominus_3 - fw\right]$
- Define $\mu_B^3 = w + z$
- Then $\left[\ominus_3 - f\mu_B^3\right] = \left[\ominus_3 - f\left(w+z\right)\right]$

$$= \left[\ominus_3 - f\left(w\right) - \left(\ominus_3 - fw + \nabla u\right)\right] = [0]$$

- Choose f_3 such that $\nabla f_3 = \ominus_3 - f\mu_B^3$

Proof of Theorem 1 (continued)

$$\begin{array}{ccccc}
\mu_B\left(\mu_B \otimes 1 - 1 \otimes \mu_B\right) & \longleftarrow & w & & z \\
\downarrow \bar{f} & & \downarrow & & \downarrow \\
f\mu_B\left(\mu_B \otimes 1 - 1 \otimes \mu_B\right) & \longleftarrow & \left\{ \begin{array}{l} fw \\ \ominus_3 \end{array} \right. & & \ominus_3 - fw - \nabla f_3
\end{array}$$

$$\mu_B^3 \;:=\; w + z \quad \text{and} \quad \nabla f_3 = \ominus_3 - f\mu_B^3$$

- $\left(B, d_B, \mu_B, \mu_B^3\right)$ is an A_3-algebra and $F_3 = f + f_2 + f_3$ is an A_3-map

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- (B, d_B, μ_B, μ_B^3) is an A_3-algebra and $F_3 = f + f_2 + f_3$ is an A_3-map
- Continue inductively

Generalization to A-infinity Bialgebras

The proof above extends immediately to A_∞-bialgebras

- In this case, f induces a cochain map

$$\tilde{f} : \left(Hom\left(B^{\otimes m}, B^{\otimes n}\right); \nabla_B \right) \to Hom\left(B^{\otimes m}, A^{\otimes n}; \nabla\right)$$

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- **Theorem 2.** *Let A be an A_∞-bialgebra and let $f : B \to A$* be a chain map. *If $\tilde{f}$ is a quasi-isomorphism, then*

  (i) f transfers the A_∞-bialgebra structure from A to B; the induced structure on B is unique up to automorphism

  (ii) There is a map $F : B \Rightarrow A$ of A_∞-bialgebras *extending f*

Example 2: Transferring A-infinity Bialgebra Structure

Transfer is controlled by a new family of polyhedra $\{JJ_{m,n}\}_{m,n\geq 1}$ of which $JJ_{n,1} = JJ_{1,n} = J_n$

- Let (M, d) be as in Example 1

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- Let (M, d) be as in Example 1
- Extend primitive Δ on M to $T^a M$ as an algebra map
- d is a coderivation of Δ since $T^a M$ is primitively generated
- $(T^a M, d, \Delta)$ is a DG bialgebra

Example 2 (continued)

- $A = T^a M \,/ \big(a_2^2 + a_4,\ a_4 a_3 + a_3 a_4,\ (a_2 a_3 + a_3 a_2)^2\big)$

Example 2 (continued)

- $A = T^a M \,/\, \left(a_2^2 + a_4,\ a_4 a_3 + a_3 a_4,\ (a_2 a_3 + a_3 a_2)^2\right)$
- $H = H^* (A)$

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- $H = H^*(A)$
- All A_∞-bialgebra relations hold; one holds non-trivially:

$$\Delta\mu_H^3 = \left[\mu_H(\mu_H \otimes 1) \otimes \mu_H^3 + \mu_H^3 \otimes \mu_H(1 \otimes \mu_H)\right]\sigma_{2,3}\Delta^{\otimes 3}$$

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- $(H, \mu_H, \mu_H^3, \Delta)$ is an A_∞-bialgebra and F is an A_∞-bialgebra map

Homology of Loop Spaces

- Let $\mathbf{k}$ be a field and let X be a space

Theorem 3. *$H_*(\Omega X; \mathbf{k})$ is an A_∞-bialgebra*

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- **Proof**. Choose a cycle-selection homomorphism

$$f : H_*(\Omega X; \mathbf{k}) \to C_*(\Omega X; \mathbf{k})$$

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Homology of Loop Spaces

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- Then $\xi \left(e^{n-2}\right) = \Delta^n$ and

$$\Delta^n \mu = \mu^{\otimes n} \sigma_{n,2} \left[(\xi \otimes \xi) \Delta_K \left(e^{n-2}\right) \right]$$

The End

Thank you!