L_∞ ALGEBRA REPRESENTATIONS

TOM LADA

1. Introduction

This note on L_∞ algebra representations is motivated by a problem in mathematical physics originally encountered in [1] and addressed in [3]. In classical gauge field theory, one encounters representations of Lie algebras in the guise of the Lie algebra of gauge parameters acting on the Lie module of fields for the theory. However, as in [1], these Lie structures occasionally appear up to homotopy

The definition of an L_∞ module structure appeared in [2]. We recall the classical result that if L is a Lie algebra and M is an L module, then the vector space $L \oplus M$ inherits a canonical Lie algebra structure. The main result in this note is the not surprising fact that the homotopy theoretic version of the above fact is also true. We hope that the explicit construction of the L_∞ structure on $L \oplus M$ will be of practical use in analyzing the algebra structures that arise in various gauge theories.

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2. L_∞ Structures

We work in the category of graded vector spaces over a fixed field k of characteristic zero. As is usual in this setting, the Koszul sign convention will be employed: whenever two symbols (objects or maps) of degrees p and q are commuted, a factor of $(-1)^{pq}$ is introduced. For a permutation σ acting on a string of symbols, we use $e(\sigma)$ to denote the total effect of these signs; the notation $\chi(\sigma) = (-1)^\sigma e(\sigma)$ where $(-1)^\sigma = sgn(\sigma)$ is the sign of the permutation σ will also be used. In order to minimize extraneous notation, we will denote the degree of a vector v by v itself when it will not lead to ambiguities; i.e. we will write $(-1)^v$ to mean $(-1)^{deg(v)}$.

Frequently, when one encounters homotopy algebra structures in examples, they usually manifest themselves as relations on elements in a

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graded vector space. However, in analyzing properties of such structures, it is usually more efficient to deal with compositions of the maps defining such structures, or, on occasion, the coderivations that result from such structures. Consequently, we present the following definitions from both points of view in the hope that the reader may profit from at least one.

Recall the definition of an L_∞ algebra.

Definition 1. *An L_∞ structure on a graded vector space L is a collection of linear maps $l_k : \otimes^k L \to L$ with $deg(l_k) = k-2$ which are skew symmetric in the sense that*

$$l_k(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}) = \chi(\sigma) l_k(v_1 \otimes \cdots \otimes v_k)$$

and satisfy the generalized Jacobi identity

$$\sum_{i+j=n+1} \sum_{\sigma} e(\sigma)(-1)^{\sigma}(-1)^{i(j-1)} l_j(l_i(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)}) \otimes \cdots \otimes v_{\sigma(n)}) = 0$$

where the sum is taken over all $(i, n-i)$ unshuffles σ.

The generalized Jacobi identity in the definition above may be written more succinctly as

$$\sum_{i+j=n+1} (-1)^{i(j-1)} l_j(l_i \otimes 1)\hat{\sigma} = 0 \tag{1}$$

where $\hat{\sigma} = \sum sgn(\sigma)\sigma$ and the sum is taken over all $(i, n-i)$ unshuffles.

We note that L_∞ algebras are also known as "sh Lie" or strongly homotopy Lie algebras. Also, if one wishes to work with cochain complexes rather than with chain complexes, the definition above remains the same except that the degree of each l_k is required to be $2-k$.

Recall that an L_∞ algebra structure on the differential graded vector space L may be described by a coderivation D of degree $+1$ on the cofree commutative coalgebra $\wedge^* \uparrow L$ with $D^2 = 0$. Here, $\uparrow$ refers to the suspension map of the graded vector space; i.e. $\uparrow L$ is the graded vector space with $(\uparrow L)_n = L_{n-1}$. The sign $(-1)^{i(j-1)}$ in the definition above results from applying the standard sign convention to the commutation of the maps l_n and the iterated suspension in the construction of D from the l_n's. These details may be found in [2].

The next definition is a rephrasing of the definition of a left module over L_∞ algebras that appeared in [2].

Definition 2. *Let $L = (L, l_k)$ be an L_∞ algebra and let M be a differential graded vector space with differential denoted by k_1. Then a left L-module structure on M is a collection of skew symmetric linear* *maps of degree* $n-2$,

$$k_n : \otimes^{n-1} L \otimes M \to M,$$

such that

$$\sum_{p+q=n+1} \sum_{\sigma(n)=n} \chi(\sigma)(-1)^{p(q-1)} k_q(l_p(\xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(p)}) \otimes \xi_{\sigma(p+1)} \otimes \cdots \otimes \xi_{\sigma(n)})$$

$$+ \sum_{p+q=n+1} \sum_{\sigma(p)=n} \chi(\sigma)(-1)^{p(q-1)}(-1)^{q-1}(-1)^{(p+\sum_{k=1}^{p}\xi_{\sigma(k)})\sum_{k=p+1}^{n}\xi_{\sigma(k)}}$$

$$k_q(\xi_{\sigma(p+1)} \otimes \cdots \otimes \xi_{\sigma(n)} \otimes k_p(\xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(p)})) = 0$$

where σ *ranges over all* $(p, n-p)$ *unshuffles,* $\xi_1, \cdots, \xi_{n-1} \in L$*, and* $\xi_n \in M$*.*

Again, we rewrite the defining relation in the definition above without elements as the sum over $p + q = n + 1$ of

$$(2) \quad \sum_{\sigma(n)=n} k_q(l_p \otimes 1) sgn(\sigma)\sigma + \sum_{\sigma(p)=n} (-1)^{q-1} k_q \tau_1 (k_p \otimes 1) sgn(\sigma)\sigma = 0.$$

Also, τ_1 is the cyclic permutation $(1 \quad q \quad q-1 \ldots 2)$ and $(-1)^{q-1}$ is the sign of τ_1. More generally, when permuting n symbols, we will make use of the cyclic permutation $\tau_i = (i \quad n \quad n-1 \quad \ldots i+1)$ along with its sign $(-1)^{n-i}$ in the next section.

Of course, the fundamental example of an L_∞ module structure occurs in the situation when $L = M$ and each $k_i = l_i$, i.e. L is an L_∞ module over itself.

3. Main Result

In the classical Lie case, if M is a Lie module over the Lie algebra L, then the vector space $L \oplus M$ possesses a canonical Lie algebra structure. We now show that exactly the same result holds for the L_∞ case.

Theorem 1. *Suppose that* (L, l_k) *is an* L_∞ *algebra and that* (M, k_n) *is a left* L_∞ *module in the sense of the previous definition. Then the graded vector space* $L \oplus M$ *has an* L_∞ *algebra structure given by*

$$j_n\{(v_1, m_1) \otimes \cdots \otimes (v_n, m_n)\} =$$

$$(l_n(v_1 \otimes \cdots \otimes v_n), \sum_{i=1}^{n} (-1)^{n-i}(-1)^{m_i \sum_{k=i+1}^{n} v_k} k_n(v_1 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes v_n, m_i)).$$

where $\hat{v}_i$ *means omit* v_i*. Again, without elements, we write*

$$j_n = (l_n \pi_1^n, \sum_{i=1}^{n} (-1)^{n-i} k_n (\pi_1^{n-1} \otimes \pi_2) \tau_i).$$

Proof. We must show that the collection of the j_n's satisfies the relations in Definition 1. We first check each coordinate map for skew symmetry.

$$\pi_1 j_n \sigma = l_n \pi_1^n \sigma = l_n \sigma \pi_1^n = sgn(\sigma) l_n \pi_1^n = \pi_1 sgn(\sigma) j_n$$

where we used the fact that each l_n is skew symmetric.

For the second coordinate, we have

$$\pi_2 j_n \sigma = \sum_{i=1}^{n} (-1)^{n-i} k_n (\pi_1^{n-1} \otimes \pi_2) \tau_i \sigma = \sum_{i=1}^{n} (-1)^{n-i} k_n (\pi_1^{n-1} \otimes \pi_2) \sigma' \tau_{\sigma^{-1}(i)}$$

where σ' is the unique permutation such that $\sigma'(n) = n$ and $\tau_i \sigma = \sigma' \tau_{\sigma^{-1}(i)}$. Consequently,

$$\pi_2 j_n \sigma = \sum_{i=1}^{n} (-1)^{n-i} k_n \sigma' (\pi_1^{n-1} \otimes \pi_2) \tau_{\sigma^{-1}(i)}$$

$$= \sum_{i=1}^{n} sgn(\sigma) (-1)^{n-\sigma^{-1}(i)} k_n (\pi_1^{n-1} \otimes \pi_2) \tau_{\sigma^{-1}(i)}$$

$$= \sum_{i=1}^{n} sgn(\sigma) (-1)^{n-i} k_n (\pi_1^{n-1} \otimes \pi_2) \tau_i = \pi_2 sgn(\sigma) j_n.$$

Here we used the fact that each k_n is skew symmetric and that the permutation σ' commutes with the map $\pi_1^{n-1} \otimes \pi_2$.

We next show that the j_n's satisfy equation 1, i.e. that

$$\sum_{p+q=n+1} (-1)^{p(q-1)} j_q (j_p \otimes 1) \hat{\sigma} = 0.$$

The first coordinate of this composition is equal to

$$\sum_{p+q=n+1} (-1)^{p(q-1)} l_q \pi_1^q (l_p \pi_1^p \otimes 1^{n-p}) \hat{\sigma}$$

which is equal to 0 because the l_n's form the L_∞ algebra structure on L.

For the second coordinate, we rewrite the composition

$$(\sum_{j=1}^{q} (-1)^{n-p+1-j} k_q (\pi_1^{q-1} \otimes \pi_2) \tau_j) \circ (l_p \pi_1^p, \sum_{i=1}^{p} (-1)^{p-i} k_p (\pi_1^{p-i} \otimes \pi_2) \tau_i \otimes 1^{n-p}) \hat{\sigma}.$$

as

$$\{(-1)^{n-p} k_q (\pi_1^{q-1} \otimes \pi_2) \tau_1 (\sum_{i=1}^{p} (-1)^{p-i} k_p (\pi_1^{p-i} \otimes \pi_2) \tau_i \otimes 1^{n-p})$$

$$+\sum_{i=p+1}^{n}(-1)^{n-i}k_q(\pi_1^{q-1}\otimes\pi_2)\tau_i(l_p\pi_1^p\otimes 1^{n-p})\}\hat{\sigma}.$$

Note that the index $j=2,\ldots,q$ in the second summand is renamed as $i=p+1,\ldots n$. Also, the first summand holds for the unshuffles σ with $\sigma(p)=n$ and the second summand holds for the unshuffles σ with $\sigma(n)=n$. Next we fix r with $1\leq r\leq n$ and consider the unshuffles with $\sigma(i)=r$. When $i\leq p$, we are in the case where $\sigma(p)=n$ and the cyclic permutation τ_i acts on the first p symbols. Let σ' be the unique unshuffle with $\tau_i\sigma=\sigma'\tau_r$ and $\sigma'(j)=\sigma(j), j>p$ and $\sigma'=r$. We have that $sgn(\tau_i)sgn(\sigma)=sgn(\sigma')sgn(\tau_r)$ or $(-1)^{p-i}sgn(\sigma)=sgn(\sigma')(-1)^{n-r}$.

When $i>p$, we have $\sigma(n)=n$ and τ_i is as previously defined. Let σ' be the unique unshuffle with $\sigma'(j)=\sigma(j)$ for $j\leq p$, $\sigma'(n)=r$ and $\tau_i\sigma=\sigma'\tau_r$. Here, we have $(-1)^{n-i}sgn(\sigma)=sgn(\sigma')(-1)^{n-r}$.

The first summand above may now be written as

$$(-1)^{n-p}(-1)^{n-r}k_q\tau_1(k_p\otimes 1)\sigma'\tau_r$$

with $\sigma'(p)=n$. Also, (-1)$^{n-p}=(-1)^{q-1}$ because $p+q=n+1$.

The second summand may be written as

$$(-1)^{n-i}(-1)^{n-i}(-1)^{n-r}k_q(l_p\otimes 1)\sigma'\tau_r.$$

So for each fixed r, we have Equation 2 following the permutation τ_r and multiplied by the coefficient $(-1)^{n-r}$, each of which is equal to 0 because M is an L module.

Consequently, the L_∞ algebra relations are satisfied.

□

4. Remarks

1. A straightforward calculation will show that if L is a Lie algebra and M is an L-module, then the structure maps constructed in the previous section, j_q, for $q>2$, may be taken to be zero. Additionally, the bracket given by j_2 is the usual Lie bracket for $L\oplus M$, namely, $[(l_1,m_1),(l_2,m_2)]=([l_1,l_2],l_1\cdot m_2-l_2\cdot m_1)$.

2. In [3], an L_∞ structure is constructed on a vector space direct sum $L\oplus M$ using assumptions in [1]. In this context, L is in fact the vector space of gauge parameters and M is the vector space of fields for these particular field theories. However, in contrast to the result in the previous section, L itself is not an L_∞ algebra with M a module over L. Such structures will be analyzed in future work.

3. Of course, the L_∞ structure on $L\oplus M$ may be described in terms of a coderivation D of degree +1 on the cofree cocommutative coalgebra

$\wedge^* \uparrow (L \oplus M)$ with $D^2 = 0$. For a direct construction of D in terms of j_q, see [2].

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Department of Mathematics, North Carolina State University, Raleigh NC 27695

E-mail address: lada@math.ncsu.edu