SYMMETRIC BRACE ALGEBRAS

TOM LADA AND MARTIN MARKL

Abstract. We develop a symmetric analog of brace algebras and discuss the relation of such algebras to L_∞-algebras. We give an alternate proof that the category of symmetric brace algebras is isomorphic to the category of pre-Lie algebras. As an application, symmetric braces are used to describe transfers of strongly homotopy structures. We then explain how these symmetric brace algebras may be used to examine the L_∞-algebras that result from a particular gauge theory for massless particles of high spin.

Introduction

The interplay between homological algebra and mathematical physics provides a vast arena for research in both subjects. In algebra, the construction of multilinear operations, called braces, on the Hochschild complex of an associative algebra leads to the definition of a brace algebra on a graded vector space [9] and to subsequent applications in topological field theory [10], as well as in a particular solution to the Deligne conjecture [7, 6]. In this note, we develop the idea of a *symmetric brace algebra* in which the brace operations will possess the property of graded symmetry. Multilinear operations on the space of anti-symmetric maps on a graded vector space, e.g. the Chevalley-Eilenberg complex of a Lie algebra, provides a motivating example for this concept. Just as A_∞- algebraic relations may be neatly encoded in the brace algebra context, L_∞-algebra data is given by symmetric brace algebra relations.

Both brace and symmetric brace algebra structures yield pre-Lie algebra structures on the underlying vector space. However, a major difference is that a pre-Lie algebra may be used to define a symmetric brace algebra structure [8]. We provide a conceptual proof of this fact and along the way exhibit a construction of a model of a free symmetric algebra.

As a first application of symmetric braces, suppose that we are given chain complexes (V, ∂_V), (W, ∂_W) and chain maps $f : (V, \partial_V) \to (W, \partial_W)$, $g : (W, \partial_W) \to (V, \partial_V)$ such that the composition gf is chain homotopic to the identity $1\!\!1_V : V \to V$, via a chain-homotopy h. If V possesses an A_∞- structure or an L_∞-structure, we show that braces and symmetric braces respectively, may be used to transfer these homotopy structures to the space W.

As another application of symmetric braces, we recall some work of Berends, Burgers, and van Dam in mathematical physics that generated algebraic data describing interactions

The research of the first author was supported in part by NSF grant INT-0203119.
The research of the second author was supported by grant MŠMT ME 603 and by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z10190503.

between massless particles of high spin [1]. In [4] it was shown that these algebraic relations may be reformulated as giving the graded vector space of gauge parameters together with the fields the structure of an L_∞-algebra. In this note, we show that this L_∞-structure may be conveniently analyzed using the language of symmetric brace algebras. This places the problem into a more general algebraic context which may perhaps be applicable to other similar problems as well.

These applications demonstrate that even though the category of symmetric brace algebras is isomorphic to the category of pre-Lie algebras, symmetric braces may provide a more manageable setting for calculations than pre-Lie operations.

Each (nonsymmetric) brace algebra defines a symmetric one by an obvious symmetrization, but not every symmetric algebra is the symmetrization of a nonsymmetric one. In this sense, nonsymmetric braces are more special than symmetric ones. While symmetric braces can be expected to exist on the operadic cohomology complex of any algebra over a Koszul quadratic operad, nonsymmetric braces exist only for algebras over non-Σ operads, compare also the remarks in [14, II.3.9].

We recall, in Section 1, the definition of brace algebras and develop our concept of symmetric brace algebras. We discuss the relation of such algebras to A_∞ and to L_∞-algebras. Several general properties of symmetric brace algebras are also given here. Section 2 contains a proof that the category of symmetric brace algebras is isomorphic to the category of pre-Lie algebras. The transfer of strongly homotopy structures appears in Section 3. In Section 4, we review the algebraic background of the physics problem as formulated in [1, 4]. We place this problem into the context of symmetric brace algebras and identify several relations in this algebra that yield the same L_∞-structure that is found in [4].

1. Symmetric Brace Algebras

We begin by introducing the following necessary technical notions [11, page 2148]. For graded indeterminates $x_1, \ldots, x_n$ and a permutation $\sigma \in \Sigma_n$ define the *Koszul sign* $\epsilon = \epsilon(\sigma; x_1, \ldots, x_n)$ by

$$x_1 \wedge \cdots \wedge x_n = \epsilon(\sigma; x_1, \ldots, x_n) \cdot x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)},$$

which has to be satisfied in the free graded commutative algebra $\bigwedge(x_1, \ldots, x_n)$. We will need also the *antisymmetric Koszul sign*

$$\chi = \chi(\sigma; x_1, \ldots, x_n) := \mathrm{sgn}(\sigma) \cdot \epsilon(\sigma; x_1, \ldots, x_n).$$

Let us recall the definition of a (non-symmetric) brace algebra as given in [6].

Definition 1. *A brace algebra is a graded vector space* U *together with a collection of degree* 0 *multilinear braces* $x, x_1, \dots, x_n \longmapsto x\{x_1, \dots, x_n\}$ *that satisfy the identities*

$$x\{\ \} = x$$

and

$$x\{x_1, ..., x_m\}\{y_1, ..., y_n\} = \sum \epsilon \cdot x\{y_1, ..., y_{i_1}, x_1\{y_{i_1+1}, ..., y_{j_1}\}, y_{j_1+1}, ..., y_{i_m}, x_m\{y_{i_m+1}, ..., y_{j_m}\}, y_{j_m+1}, ..., y_n\},$$

where the sum is taken over all sequences $0 \le i_1 \le j_1 \le \cdots \le i_m \le j_m \le n$ *and where* ϵ *is the Koszul sign of the permutation*

$$(x_1, ..., x_m, y_1, ..., y_n) \mapsto (y_1, ..., y_{i_1}, x_1, y_{i_1+1}, ..., y_{j_1}, y_{j_1+1}, ..., y_{i_m}, x_m, y_{i_m+1}, ..., y_{j_m}, y_{j_m+1}, ..., y_n)$$

of elements of U.

In [6], two gradings of the underlying vector space, $\deg(x)$ and $|x|$, related by $|x| = \deg(x) - 1$, were used. The brace $x\{x_1, \dots, x_n\}$ there was of degree $-n$ with respect to the $\deg(-)$-grading and of degree 0 with respect to the $|-|$-grading. When we refer to [6], we always consider the underlying vector space graded with the $|-|$-grading. Then all braces will be degree zero maps, as assumed in the above definition. We now consider a symmetric version of the brace algebra.

Definition 2. *A symmetric brace algebra is a graded vector space* B *together with a collection of degree* 0 *multilinear braces* $x\langle x_1, \dots, x_n\rangle$ *that are graded symmetric in* $x_1, \dots, x_n$ *and satisfy the identities*

$$x\langle\ \rangle = x$$

and

$$x\langle x_1, \dots, x_m\rangle\langle y_1, \dots, y_n\rangle = \sum \epsilon \cdot x\langle x_1\langle y_{i^1_1}, \dots, y_{i^1_{t_1}}\rangle, x_2\langle y_{i^2_1}, \dots, y_{i^2_{t_2}}\rangle, \dots, x_m\langle y_{i^m_1}, \dots, y_{i^m_{t_m}}\rangle, y_{i^{m+1}_1}, \dots, y_{i^{m+1}_{t_{m+1}}}\rangle \tag{1}$$

where the sum is taken over all unshuffle sequences

$$i^1_1 < \cdots < i^1_{t_1}, \dots, i^{m+1}_1 < \cdots < i^{m+1}_{t_{m+1}}$$

of $\{1, \dots, n\}$ *and where* ϵ *is the Koszul sign of the permutation*

$$(x_1, ..., x_m, y_1, ..., y_n) \longmapsto (x_1, y_{i^1_1}, ..., y_{i^1_{t_1}}, x_2, y_{i^2_1}, ..., y_{i^2_{t_2}}, ..., x_m, y_{i^m_1}, ..., y_{i^m_{t_m}}, y_{i^{m+1}_1}, ..., y_{i^{m+1}_{t_{m+1}}})$$

of elements of B.

Exercise 3. For elements x, y of an arbitrary symmetric brace algebra B, put

$$x \circ y := x\langle y\rangle.$$

Prove that then $(B, \circ)$ is a graded right pre-Lie algebra in the sense of [5, Section 2], therefore $[x, y] := x \circ y - (-1)^{|x||y|} y \circ x$ defines a graded Lie algebra structure on B. This result should be compared to a similar result for non-symmetric brace algebras in [6, p. 143].

This Lie algebra structure corresponds to the standard anti-commutator Lie algebra on the space of coderivations of the cocommutative coassociative coalgebra ${}^c\!\bigwedge(W)$ [11, page 2150] cogenerated by the suspension $W := \uparrow V$ of V. Compare also remarks in [14, II.3.9].

Remark 4. J.-M. Oudom and D. Guin proved in [8] that higher brackets $x\langle x_1, \ldots, x_n\rangle$ of an arbitrary symmetric brace algebra are, for $n \geq 2$, determined by the 'pre-Lie part' $x \circ y = x\langle y\rangle$, introduced in Exercise 3. For instance, axiom (1) implies that $x\langle x_1, x_2\rangle$ can be expressed as

$$x\langle x_1, x_2\rangle = x\langle x_1\rangle\langle x_2\rangle - x\langle x_1\langle x_2\rangle\rangle = (x \circ x_1) \circ x_2 - x \circ (x_1 \circ x_2).$$

The same axiom applied on $x\langle x_1, \ldots, x_{n-1}\rangle\langle x_n\rangle$ can then be clearly interpreted as an inductive rule defining $x\langle x_1, \ldots, x_n\rangle$ in terms of $x\langle x_1, \ldots, x_k\rangle$, with $k < n$.

They also proved that *an arbitrary* pre-Lie algebra determines in this way a symmetric brace algebra, which would mean that the category of symmetric brace algebras *is isomorphic* to the category of pre-Lie algebras. Let us observe that the proof of this statement is not obvious. First, axiom (1) interpreted as an inductive rule is 'overdetermined.' For example, $x\langle x_1, x_2, x_3\rangle$ can be expressed both from (1) applied to $x\langle x_1, x_2\rangle\langle x_3\rangle$ and also from (1) applied to $x\langle x_1\rangle\langle x_2, x_3\rangle$, and it is not obvious that the results are the same. Second, even if the braces are well-defined, it is not clear that they satisfy the axioms of brace algebras, including the graded symmetry.

Example 5. Just as there is a (nonsymmetric) brace algebra structure on the graded vector space $\bigoplus_{k\geq 1} Hom(V^{\otimes k}, V)$ (see [6]), the basic example of a symmetric brace algebra is provided by the space of antisymmetric (another terminology: alternating) maps, $\bigoplus_{k\geq 1} Hom(V^{\otimes k}, V)^{as}$. More precisely, let $B(V) = B_*(V)$ be the graded vector space with components

$$B_s(V) := \bigoplus_{p-k+1=s} Hom(V^{\otimes k}, V)^{as}_p,$$

where $Hom(V^{\otimes k}, V)^{as}_p$ denotes the space of k-multilinear maps of degree p that are antisymmetric (or alternating) in the sense that

$$f(v_1, \ldots, v_i, v_{i+1}, \ldots, v_k) = -(-1)^{\deg(v_i)\deg(v_{i+1})} f(v_1, \ldots, v_{i+1}, v_i, \ldots, v_k),$$

for any $v_1, \ldots, v_i, v_{i+1}, \ldots, v_k \in V$ and $1 \leq i \leq k-1$.

Given $f \in Hom(V^{\otimes k}, V)^{as}_p$ and $g_i \in Hom(V^{\otimes a_i}, V)^{as}_{q_i}$, $1 \leq i \leq n$, define the symmetric brace $f\langle g_1, \ldots, g_n\rangle \in Hom(V^{\otimes r}, V)^{as}_{p+q_1+\cdots+q_n}$, where $r := a_1 + \cdots + a_n + k - n$, by

$$f\langle g_1, \ldots, g_n\rangle(v_1, \ldots, v_r) := \sum (-1)^{\delta} \chi \cdot f(g_1 \otimes \cdots \otimes g_n \otimes \mathbb{1}^{\otimes k-n})(v_{i_1}, \ldots, v_{i_r}) \quad (2)$$

with the summation taken over all unshuffles

$$i_1 < \cdots < i_{a_1}, i_{a_1+1} < \ldots < i_{a_1+a_2}, \ldots, i_{a_1+\cdots+a_k+1} < \cdots < i_r,$$

of elements of V, where χ is the antisymmetric Koszul sign of the permutation

$$(v_1, \ldots, v_r) \longmapsto (v_{i_1}, \ldots, v_{i_r})$$

and

$$\begin{aligned}\delta &= (k-1)q_1 + (k-2+a_1)q_2 + \cdots + (k-n+a_1+\cdots+a_{n-1})q_n \\ &\quad + \textstyle\sum_{1\le i<j\le n} a_i a_j + (n-1)a_1 + (n-2)a_2 + \cdots + a_{n-1}.\end{aligned}$$

Exercise 6. Just as an A_∞-structure on V may be described by the brace algebra relation $\mu\{\mu\} = 0$, with $\mu = \mu_1 + \mu_2 + \cdots$, $\mu_k \in \mathit{Hom}(V^{\otimes k}, V)_{k-2}$, an L_∞-algebra structure on V can be described by the symmetric brace algebra relation $l\langle l\rangle = 0$; here $l = l_1 + l_2 + \cdots$ where each $l_k \in \mathit{Hom}(V^{\otimes k}, V)^{as}_{k-2} \in B_{-1}(V)$. Strictly speaking, elements μ and l belong to the completions $\prod_{k\ge1} \mathit{Hom}(V^{\otimes k}, V)_{k-2}$ and $\prod_{k\ge1} \mathit{Hom}(V^{\otimes k}, V)^{as}_{k-2}$ of the underlying graded vector spaces, but it is immediately clear that the above statements make sense also in this more general setup.

Exercise 7. As a very particular case of Exercise 6, each Lie algebra structure on V determines an element $l = l_2 \in \mathit{Hom}(V^{\otimes 2}, V)^{as}_0 \subset B_{-1}(V)$ such that $l\langle l\rangle = 0$. Prove that then the formulas

$$\partial f := l\langle f\rangle - (-1)^{|f|} f\langle l\rangle \text{ and } \{f, g\} := l\langle f, g\rangle$$

define on $B(V)$ a differential graded Lie algebra, with a degree -1 bracket $\{-,-\}$ and degree -1 differential ∂. Verify also the formula

$$\{f, g\} = \partial f \circ g + (-1)^{|f|} f \circ \partial g - \partial(f \circ g)$$

which shows that that the bracket $\{-,-\}$ is actually *cohomologous to zero*, with the chain homotopy given by $f \circ g$.

It is easy to see that, in the situation of Exercise 7, the bigraded complex

$$CE^*(V) := (B_{1-*}(V), \partial)$$

is the standard Chevalley-Eilenberg complex of the graded Lie algebra (V, l). Brackets $[-,-]$ and $\{-,-\}$, introduced in Exercises 3 and 7, induce on $CE^*(V)$ two Lie brackets, which we denote again by $[-,-]$ and $\{-,-\}$, of degrees -1 and 0 respectively. The first bracket $[-,-]$, whose definition does not involve l, is the intrinsic bracket considered, for example, in [15]. The second bracket $\{-,-\}$ should be considered as an analog of the $\smile$-product on Hochschild cochain complex of an associative algebra, see [6]. The above calculation shows that this bracket is homologically trivial, therefore one could not expect the Chevalley-Eilenberg cohomology of Lie algebras to have a similar rich structure as the Hochschild cohomology of associative algebras.

The relationship between brace algebras and symmetric brace algebras may be summarized by the following theorems.

Theorem 8. *Let $-\{-,\cdots,-\}$ be a (non-symmetric) brace algebra structure on a graded vector space U. Then*

$$f\langle g_1,\ldots,g_n\rangle := \sum_{\sigma\in\Sigma_n} \epsilon\cdot f\{g_{\sigma(1)},\ldots,g_{\sigma(n)}\},$$

where ϵ denotes the Koszul sign of the permutation

$$(g_1,\ldots,g_n) \longmapsto (g_{\sigma(1)},\ldots,g_{\sigma(n)}),$$

gives U the structure of a symmetric brace algebra.

Now let $as(f)$ denote the anti-symmetrization

$$as(f)(v_1,\ldots,v_k) := \sum_{\sigma\in\Sigma_k} \chi\cdot f(v_{\sigma(1)},\ldots,v_{\sigma(k)}) \tag{3}$$

of a linear map $f : V^{\otimes k} \to V$.

Theorem 9. *The symmetrization of the (non-symmetric) braces on $\bigoplus_{k\geq 1} Hom(V^{\otimes k},V)$ constructed in [6] coincides with the symmetric braces (2). By this we mean that, for each $f,g_1,\ldots,g_n \in Hom(V^{\otimes *},V)$,*

$$\sum_{\sigma\in\Sigma_n} \epsilon\cdot as(f\{g_{\sigma(1)},\ldots,g_{\sigma(n)}\}) = as(f)\langle as(g_1),\ldots,as(g_n)\rangle,$$

where ϵ is the Koszul sign of the permutation

$$(g_1,\ldots,g_n) \longmapsto (g_{\sigma(1)},\ldots,g_{\sigma(n)}).$$

The proofs of these theorems will appear in [3]. Theorems 8 and 9 can be combined into:

Corollary 10. *Let us consider the space $\bigoplus_{k\geq 1} Hom(V^{\otimes k},V)$ with the symmetric brace algebra structure given by the symmetrization of the (non-symmetric) brace algebra of* [6]. *Consider also the space of anti-symmetric maps $\bigoplus_{k\geq 1} Hom(V^{\otimes k},V)^{as}$, with the symmetric braces (2). Then the anti-symmetrization*

$$as : \bigoplus_{k\geq 1} Hom(V^{\otimes k},V) \to \bigoplus_{k\geq 1} Hom(V^{\otimes k},V)^{as}$$

defined in (3) is a homomorphism of symmetric brace algebras.

As a corollary to Theorem 9, we obtain Theorem 3.1 of [11]:

Corollary 11. *The anti-symmetrization $l := as(\mu)$ of an A_∞-structure μ yields an L_∞-structure.*

Proof. The proof immediately follows from $as(\mu\{\mu\}) = as(\mu)\langle as(\mu)\rangle = l\langle l\rangle$. □

Remark 12. There are two degree conventions for L_∞-algebras. Under the first convention used in [11], structure operations are graded antisymmetric maps $l_k : V^{\otimes k} \to V$ of degree $k-2$ which can be pieced together into a degree -1 coderivation of ${}^c\!\bigwedge(\uparrow V)$. Under the second convention, structure operations are maps $l_k : V^{\otimes k} \to V$ of degree $2-k$ which can be encoded by a degree $+1$ coderivation of ${}^c\!\bigwedge(\downarrow V)$. These conventions differ by the "flip" $V_k \mapsto V_{-k}$ of the underlying graded vector space V. In this paper, we use the first convention.

As we saw in the second half of this section, symmetric brace algebras may be interpreted as a tool formalizing compositions of anti-symmetric maps. In the remaining sections, we try to convince the reader that this formalization can be used for concrete calculations.

2. Equivalence between symmetric brace algebras and pre-Lie algebras

In this section we show that the category of symmetric brace algebras is isomorphic to the category of pre-Lie algebras (Proposition 17). As we already observed in Remark 4, this statement was proved in [8], but we attempt to give a more direct and conceptual proof. We show that the free symmetric brace algebra $\mathrm{SB}(X)$ generated by a set X is functorially isomorphic to the free pre-Lie algebra $\mathrm{PL}(X)$ generated by the same set. This, by arguments analyzed for example in [14, Section II.1.4], means that the operad $\mathcal{SB}$ describing symmetric brace algebras is isomorphic to the operad $p\mathcal{L}ie$ for pre-Lie algebras, which tautologically implies that the corresponding *categories of algebras* are isomorphic. Since both operads $\mathcal{SB}$ and $p\mathcal{L}ie$ are concentrated in degree zero, it is enough to consider in this section only non-graded objects, avoiding thus the sign issue completely.

Let us start with some necessary technicalities. By a *rooted tree* we understand a nonempty contractible graph with a distinguished vertex called the *root*. We denote $\mathcal{T}ree$ the set of all rooted trees. For $T \in \mathcal{T}ree$, let $Vert(T)$ (resp. $Edg(T)$) denote the set of vertices (resp. edges) of T. We will call the unique rooted tree with one vertex r (which is also its root) the *singleton* and denote it $\{r\}$.

Let $S, S_1, \ldots, S_n \in \mathcal{T}ree$, $n \geq 1$, be rooted trees and let $\mathbf{v} : \{S_1, \ldots, S_n\} \to Vert(S)$ be a map that assigns to each element of the set $\{S_1, \ldots, S_n\}$ a vertex of S. Let $S\{S_1, \ldots, S_n\}_{\mathbf{v}} \in \mathcal{T}ree$ be the tree obtained by connecting, for $1 \leq i \leq n$, the root of S_i to the vertex $\mathbf{v}(S_i) \in Vert(S)$. Therefore

$$\begin{aligned} Vert(S\{S_1, \ldots, S_n\}_{\mathbf{v}}) &= Vert(S) \sqcup Vert(S_1) \sqcup \cdots \sqcup Vert(S_n) \text{ and} \\ Edg(S\{S_1, \ldots, S_n\}_{\mathbf{v}}) &= Edg(S) \sqcup Edg(S_1) \sqcup \cdots \sqcup Edg(S_n) \sqcup \{e_1, \ldots, e_n\}, \end{aligned}$$

where e_i is a new edge that joints the vertex $\mathbf{v}(S_i) \in Vert(S)$ with the root of the tree S_i. Let us emphasize that in the notation $S\{S_1, \ldots, S_n\}_{\mathbf{v}}$ the curly braces indicate that the construction depends only on the *set* of trees $S_1, \ldots, S_n$ not on their relative order and have therefore nothing in common with the non-symmetric braces considered in Section 1.

Example 13. Let T be a tree with at least two vertices and let $r \in \mathit{Vert}(T)$ be its root. Then there exist a unique set $\{T_1, \ldots, T_m\}$ of rooted trees such that

$$T = \{r\}\{T_1, \ldots, T_m\}_{\mathbf{v}},$$

where $\mathbf{v} : \{T_1, \ldots, T_m\} \to \{r\}$ is the unique set map that sends each T_i to the root r of the singleton $\{r\}$. Since $\mathbf{v}$ carries no information, we will drop it from the notation and write simply $T = \{r\}\{T_1, \ldots, T_m\}$.

Let B be an arbitrary symmetric brace algebra, $T \in \mathcal{T}\mathit{ree}$ a rooted tree and $\mathbf{b} : \mathit{Vert}(T) \to B$ a set map. We may interpert $\mathbf{b}$ as a *decoration* of the vertices of T with elements of B. For such a decoration we define the element $T(\mathbf{b}) \in B$ inductively as follows.

If $T = \{r\}$ is the singleton, we put $T(\mathbf{b}) := \mathbf{b}(r) \in B$. If T has at least two vertices, it decomposes as in Example 13 into $T = \{r\}\{T_1, \ldots, T_m\}$. Using the restrictions $\mathbf{b}_j := \mathbf{b}|_{\mathit{Vert}(T_j)} : \mathit{Vert}(T_j) \to B$, $1 \leq j \leq m$, we define

$$T(\mathbf{b}) = \{r\}\{T_1, \ldots, T_m\}(\mathbf{b}) := \mathbf{b}(r)\langle T_1(\mathbf{b}_1), \ldots, T_m(\mathbf{b}_m)\rangle \in B. \tag{4}$$

In the above display, $-\langle -, \ldots, -\rangle$ denotes the symmetric brace of B. Since each T_j, $1 \leq j \leq m$, has strictly fewer vertices than T, the elements $T_j(\mathbf{b}_j) \in B$ have already been defined by induction.

In the following proposition, which is the main technical result of this section, we formulate an extension of axiom (1) of Definition 2.

Proposition 14. *Let B be an arbitrary brace algebra, $S, S_1, \ldots, S_n \in \mathcal{T}\mathit{ree}$ rooted trees and $\mathbf{c} : \mathit{Vert}(S) \to B, \mathbf{c}_i : \mathit{Vert}(S_i) \to B$, $1 \leq i \leq n$, decorations of vertices. Then*

$$S(\mathbf{c})\langle S_1(\mathbf{c}_1), \ldots, S_n(\mathbf{c}_n)\rangle = \sum_{\mathbf{v}:\{S_1,\ldots,S_n\}\to \mathit{Vert}(S)} S\{S_1, \ldots, S_n\}_{\mathbf{v}}(\mathbf{c} \sqcup \mathbf{c}_1 \sqcup \cdots \sqcup \mathbf{c}_n), \tag{5}$$

where

$$\mathbf{c} \sqcup \mathbf{c}_1 \sqcup \cdots \sqcup \mathbf{c}_n : \mathit{Vert}(S\{S_1, \ldots, S_n\}_{\mathbf{v}}) = \mathit{Vert}(S) \sqcup \mathit{Vert}(S_1) \sqcup \cdots \sqcup \mathit{Vert}(S_n) \to B$$

is the decoration induced in the obvious way from the decorations $\mathbf{c}, \mathbf{c}_1, \ldots, \mathbf{c}_n$.

Proof. The proof is given by induction on the number of vertices of S. If S is the singleton, then (5) follows immediately from the defining equation (4). If S has at least two vertices, then

$$S = \{r\}\{T_1, \ldots, T_m\}$$

for some $T_j \in \mathcal{T}\mathit{ree}$ as in Example 13. Let $\mathbf{b}_j := \mathbf{c}|_{\mathit{Vert}(T_j)}$, $1 \leq j \leq m$. Then the left hand side of (5) can be expanded into

$$\mathbf{c}(r)\langle T_1(\mathbf{b}_1), \ldots, T_m(\mathbf{b}_m)\rangle\langle S_1(\mathbf{c}_1), \ldots, S_n(\mathbf{c}_n)\rangle.$$

In the rest of this proof, we simplify the notation by writing $S_i(\mathbf{c})$ instead of $S_i(\mathbf{c}_i)$, $1 \leq i \leq n$, and $T_j(\mathbf{b})$ instead of $T_j(\mathbf{b}_j)$, $1 \leq j \leq m$. We believe that such a simplification will not confuse the reader. With this convention assumed, the left hand side of (5) reads

$$\mathbf{c}(r)\langle T_1(\mathbf{b}), \ldots, T_m(\mathbf{b})\rangle\langle S_1(\mathbf{c}), \ldots, S_n(\mathbf{c})\rangle.$$

Axiom (1) converts this expression into

$$\sum_{ush} \mathbf{c}(r) \left\langle T_1(\mathbf{b})\langle S_{i^1_1}(\mathbf{c}), ..., S_{i^1_{t_1}}(\mathbf{c})\rangle, \ldots, T_m(\mathbf{b})\langle S_{i^m_1}(\mathbf{c}), ..., S_{i^m_{t_m}}(\mathbf{c})\rangle, S_{i^{m+1}_1}(\mathbf{c}), \ldots, S_{i^{m+1}_{t_{m+1}}}(\mathbf{c}) \right\rangle,$$

where the summation runs over the same set of unshuffles as in (1). Since each T_j has fewer vertices than S, we may use the induction to convert the above expression into

$$\sum_{ush} \sum_{\mathbf{v}_1,\ldots\mathbf{v}_m} \mathbf{c}(r) \left\langle T_1\{S_{i^1_1}, ..., S_{i^1_{t_1}}\}_{\mathbf{v}_1}(\mathbf{b} \sqcup \mathbf{c}), ..., T_m\{S_{i^m_1}, ..., S_{i^m_{t_m}}\}_{\mathbf{v}_m}(\mathbf{b} \sqcup \mathbf{c}), S_{i^{m+1}_1}(\mathbf{c}), ..., S_{i^{m+1}_{t_{m+1}}}(\mathbf{c}) \right\rangle$$

where each $\mathbf{v}_j$ runs over all set maps $\mathbf{v}_j : \{S_{i^j_1}, \ldots, S_{i^1_{t_j}}\} \to \mathit{Vert}(T_j)$, $1 \leq j \leq m$.

Using (4), the above display can be written as

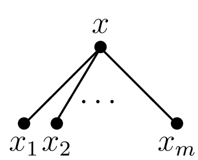
$$\sum_{ush} \sum_{\mathbf{v}_1,\ldots\mathbf{v}_m} \{r\} \left\{ T_1\{S_{i^1_1}, ..., S_{i^1_{t_1}}\}, \ldots, T_m\{S_{i^m_1}, ..., S_{i^m_{t_m}}\}, S_{i^{m+1}_1} \ldots, S_{i^{m+1}_{i_{m+1}}} \right\} (\mathbf{b} \sqcup \mathbf{c})$$

which can be easily identified with

$$\sum_{\mathbf{v}:\{S_1,\ldots,S_n\} \to \mathit{Vert}(S)} S\{S_1, \ldots, S_n\}_{\mathbf{v}}(\mathbf{c})$$

which is the right hand side of (5). □

Exercise 15. Show that axiom (1) is a particular case of (5) for S the corrola



with the root decorated by $x \in B$ and the remaining vertices by $x_1, \ldots, x_m \in B$, and S_i's the singletons with the unique vertex decorated by y_i, $1 \leq i \leq n$.

Let us describe our realization of the free symmetric brace algebra. For a set X define

$$\mathrm{SB}(X) := \mathit{Span}\{X_T;\ T \in \mathcal{T}\!\mathit{ree}\},$$

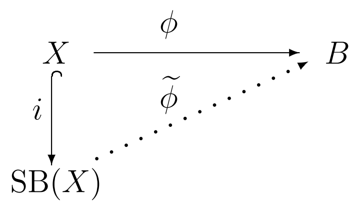
where X_T is the set of all decorations $\mathbf{x} : \mathit{Vert}(T) \to X$ of the vertices of the tree T by elements of X. Let us introduce symmetric braces on the vector space $\mathrm{SB}(X)$ as follows. For $\mathbf{x} : \mathit{Vert}(S) \to X \in X_T$ and $\mathbf{x}_i : \mathit{Vert}(S_i) \to X \in X_{S_i}$, $1 \leq i \leq n$, set

$$\mathbf{x}\langle \mathbf{x}_1, \ldots, \mathbf{x}_n\rangle := \sum_{\mathbf{v}:\{S_1,\ldots,S_n\} \to \mathit{Vert}(S)} \mathbf{x} \sqcup \mathbf{x}_1 \sqcup \cdots \sqcup \mathbf{x}_n,$$

where $\mathbf{x} \sqcup \mathbf{x}_1 \sqcup \cdots \sqcup \mathbf{x}_n : \mathit{Vert}(S\{S_1, \ldots, S_n\}_{\mathbf{v}}) \to X$ is the obvious induced decoration of $S\{S_1, \ldots, S_n\}_{\mathbf{v}}$. The canonical map $i : X \hookrightarrow \mathrm{SB}(X)$ sends $x \in X$ to the singleton $\{r\}$ decorated by x.

Theorem 16. *The object $X \hookrightarrow \mathrm{SB}(X)$ defined above is the free symmetric brace algebra on X.*

Proof. Observe first that Propostion 14 implies that $\mathrm{SB}(X)$ is indeed a symmetric brace algebra. One must show next that for an arbitrary symmetric brace algebra B and for an arbitrary set map $\phi : X \to B$ there exist a unique homomorphism $\widetilde{\phi} : \mathrm{SB}(X) \to B$ of symmetric brace algebras for which the diagram



commutes. A moment's reflection convinces us that the only possible choice is

$$\widetilde{\phi}(\mathbf{x}) := T(\phi \circ \mathbf{x}),$$

where, for $\mathbf{x} : \mathit{Vert}(T) \to X \in X_T$, $\phi \circ \mathbf{x}$ is the composition $\mathit{Vert}(T) \xrightarrow{\mathbf{x}} X \xrightarrow{\phi} B$. Such a $\widetilde{\phi} : \mathrm{SB}(X) \to B$ is, again by Proposition 14, indeed a symmetric brace algebra homomorphism. This finishes the proof. □

The following proposition is due to D. Guin and J.-M. Oudom [8].

Proposition 17. *The category of symmetric brace algebras is isomorphic to the category of pre-Lie algebras.*

Proof. It is immediate to see that the pre-Lie algebra $\mathrm{SB}(X)_{pL}$ associated to the free brace algebra $\mathrm{SB}(X)$ is canonically isomorphic to the free pre-Lie algebra $\mathrm{PL}(X)$ as described in [2]. Proposition II.1.27 of [14] implies that the corresponding operads $\mathcal{SB}$ and $p\mathcal{L}ie$ are isomorphic, and the proposition follows. □

3. Transfers of strongly homotopy structures

In this section we show how brace algebras can be used to simplify formulas for transfers of strongly homotopy structures. In [12, 13] we considered the following situation.

Situation 18. *We are given chain complexes (V, ∂_V), (W, ∂_W) and chain maps $f : (V, \partial_V) \to (W, \partial_W)$, $g : (W, \partial_W) \to (V, \partial_V)$ such that the composition gf is chain homotopic to the identity $1\!\!1_V : V \to V$, via a chain-homotopy h. In other words, g is a left homotopy inverse of f.*

We assumed that (V, ∂_V) was equipped with an A_∞-structure $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3, \ldots)$ with $\mu_1 = \partial_V$. We were looking for an A_∞-structure $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3, \ldots)$ on (W, ∂_W), with $\nu_1 = \partial_W$, such that the A_∞-structures $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ were equivalent, via a suitable extensions of the chain maps f and g, see [13, Problem 2] for a precise formulation.

We call the A_∞-structure $\boldsymbol{\nu}$ with the properties specified in the above paragraph the *transfer* of $\boldsymbol{\mu}$. The existence of transfers follows from general principles (see [12]), but in [13] we constructed such a $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3, \ldots)$ explicitly by the "Anzatz"

$$\nu_n := f \circ \boldsymbol{p}_n \circ g^{\otimes n},\ n \geq 2, \tag{6}$$

where $\boldsymbol{p}_n : V^{\otimes n} \to V$ (the *p-kernel*) was a degree $n-1$ linear map defined inductively by $\boldsymbol{p}_2 := \mu_2$ and

$$\boldsymbol{p}_n := \sum_B (-1)^{\vartheta(r_1,\ldots,r_k)} \mu_k(h \circ \boldsymbol{p}_{r_1} \otimes \cdots \otimes h \circ \boldsymbol{p}_{r_k}), \quad \text{for } n \geq 2. \tag{7}$$

In the above display we denoted

$$\vartheta(u_1, \ldots, u_s) := \sum_{1 \leq \alpha < \beta \leq s} u_\alpha(u_\beta + 1),$$

the summation was taken over

$$B := \{k, r_1, \ldots, r_k \mid 2 \leq k \leq n,\ r_1, \ldots, r_k \geq 1,\ r_1 + \cdots + r_k = n\}$$

and the formal convention that $h\boldsymbol{p}_1 = \mathbb{1}$ was assumed. We proved that the p-kernel satisfies

$$\partial(\boldsymbol{p}_n) = \sum_A (-1)^{i(l+1)+n} \boldsymbol{p}_k(\mathbb{1}^{\otimes i-1} \otimes gf \circ \boldsymbol{p}_l \otimes \mathbb{1}^{\otimes k-i}),\ n \geq 2, \tag{8}$$

where ∂ is the differential on $Hom(V^{\otimes n}, V)$ induced from ∂_V in the standard way. We then derived from (8) that the operations defined by the Anzatz (6) indeed form an A_∞-structure on (W, ∂_W).

Let us translate the above calculations into the language of brace algebras. Consider the graded vector space $End_*(V)$ defined by

$$End_s(V) := \bigoplus_{p-k+1=s} Hom(V^{\otimes k}, V)_p$$

with the (nonsymmetric) brace algebra structure introduced in [6]. As we observed in Example 5, the A_∞-structure $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3, \ldots)$ assembles into an element $\mu := \mu_1 + \mu_2 + \mu_3 + \cdots \in End_{-1}(V)$ that satisfies $\mu\{\mu\} = 0$. Let $\overline{\mu} := \mu_2 + \mu_3 + \cdots$ so that $\mu = \partial_V + \overline{\mu}$ (recall that we assumed $\mu_1 = \partial_V$). Therefore

$$\mu\{\mu\} = \partial_V\{\partial_V\} + \partial_V\{\overline{\mu}\} + \overline{\mu}\{\partial_V\} + \overline{\mu}\{\overline{\mu}\} = 0 \tag{9}$$

where, of course, $\partial_V\{\partial_V\} = 0$. Let, for $u \in End_*(V)$,

$$\partial u := \partial_V\{u\} - (-1)^{|u|} u\{\partial_V\}.$$

With this notation assumed, the defining identity (9) for A_∞-structures can be rewritten as $\partial\overline{\mu} + \overline{\mu}\{\overline{\mu}\} = 0$. The same analysis takes place also for $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3, \ldots)$, that is, $\overline{\nu} := \nu_2 + \nu_3 + \cdots \in End_{-1}(W)$ defines an A_∞-structure on (W, ∂_W) if and only if

$$\partial\overline{\nu} + \overline{\nu}\{\overline{\nu}\} = 0. \tag{10}$$

The maps $f : V \to W$ and $g : W \to V$ induce a degree zero map $\Phi : End_*(V) \to End_*(W)$ that sends $u \in Hom(V^{\otimes n}, V)$ to $f \circ u \circ g^{\otimes n}$. It is clear that $\partial\Phi = \Phi\partial$ and that

$$\Phi(u)\{\Phi(u_1), \ldots, \Phi(u_n)\} = \Phi(u\{gf \circ u_1, \ldots, gf \circ u_n\}), \tag{11}$$

for $u, u_1, \ldots, u_n \in End_*(V)$.

Let finally $\boldsymbol{p} := \boldsymbol{p}_2 + \boldsymbol{p}_3 + \cdots \in End_{-1}(V)$. In the above notation, the Anzatz (6) reads:

$$\overline{\nu} := \Phi(\boldsymbol{p}). \tag{12}$$

With a little effort, one verifies that the inductive formula (7) defining the p-kernel can be rewritten as

$$\boldsymbol{p} = \overline{\mu} + \overline{\mu}\{h \circ \boldsymbol{p}\} + \overline{\mu}\{h \circ \boldsymbol{p}, h \circ \boldsymbol{p}\} + \overline{\mu}\{h \circ \boldsymbol{p}, h \circ \boldsymbol{p}, h \circ \boldsymbol{p}\} + \cdots . \tag{13}$$

Setting formally $\mu\{1\} := \mu$, the above display can further be shortened into

$$\boldsymbol{p} = \overline{\mu}\left\{\frac{1}{1 - h \circ \boldsymbol{p}}\right\}.$$

In the same vein, formula (8) reads:

$$\partial\boldsymbol{p} + \boldsymbol{p}\{gf \circ \boldsymbol{p}\} = 0. \tag{14}$$

Let us show that (14) indeed implies the master equation (10) for $\overline{\nu}$ defined by (12). We have

$$\partial\overline{\nu} = \partial\Phi(\partial\boldsymbol{p}) = \Phi(\partial\boldsymbol{p}) = -\Phi(\boldsymbol{p}\{gf \circ \boldsymbol{p}\}) = -\Phi(\boldsymbol{p})\{\Phi(\boldsymbol{p})\} = -\overline{\nu}\{\overline{\nu}\}.$$

where we used (11) and the fact that Φ is a chain map. Therefore $\partial\overline{\nu} = -\overline{\nu}\{\overline{\nu}\}$, which is (10). We leave as an exercise to prove (14) using axioms of brace algebras only.

Let us consider the L_∞-version of the above problem. In Situation 18, we are given an L_∞-structure $\boldsymbol{l} = (l_1, l_2, l_3, \ldots)$, with $l_1 = \partial_V$, on (V, ∂_V). We are looking for its transfer $\boldsymbol{k} = (k_1, k_2, k_3, \ldots)$ onto (W, ∂_W), with $k_1 = \partial_W$.

It is clear that instead of $End_*(V)$ which helped us with the A_∞-case, we need the symmetric brace algebra $B_*(V)$ introduced in Example 5. Let $\bar{l} := l_2 + l_3 + \cdots \in B_{-1}(V)$. The following theorem gives an explicit construction of a transfer $\boldsymbol{k}$ of $\boldsymbol{l}$.

Theorem 19. *The p-kernel $\boldsymbol{p} = \boldsymbol{p}_2 + \boldsymbol{p}_3 + \cdots \in B_{-1}(V)$ defined inductively by $\boldsymbol{p}_2 := l_2$ and*

$$\boldsymbol{p} := \bar{l} + \bar{l}\langle h \circ \boldsymbol{p}\rangle + \frac{1}{2!}\bar{l}\langle h \circ \boldsymbol{p}, h \circ \boldsymbol{p}\rangle + \frac{1}{3!}\bar{l}\langle h \circ \boldsymbol{p}, h \circ \boldsymbol{p}, h \circ \boldsymbol{p}\rangle + \cdots \tag{15}$$

satisfies

$$\partial \boldsymbol{p} + \boldsymbol{p}\langle gf \circ \boldsymbol{p}\rangle = 0. \tag{16}$$

The Anzatz

$$\overline{k} := \Phi(\boldsymbol{p}) \tag{17}$$

then defines an L_∞-structure $\boldsymbol{k} = (k_1, k_2, k_3, \ldots)$, with $k_1 = \partial_W$ on (W, ∂_W), where Φ is defined in exactly the same way as in the A_∞-case.

Equation (15) is the symmetrization in the sense of Theorem 9 of (13). We will shorten it into

$$\boldsymbol{p} = \bar{l}\langle \exp(h \circ \boldsymbol{p})\rangle. \tag{18}$$

Our proof of Theorem 19 will be based on the following lemma.

Lemma 20. *Elements a, b, c, with $\deg(b) = \deg(c) = -1$, of an arbitrary symmetric brace algebra satisfy*

$$a\langle \exp c\rangle\langle b\rangle = a\langle c\langle b\rangle, \exp c\rangle + a\langle b, \exp c\rangle \tag{19}$$

and

$$a\langle b\rangle\langle \exp c\rangle = a\langle b\langle \exp c\rangle, \exp c\rangle. \tag{20}$$

Proof. The left hand side of (19) reads

$$\begin{aligned}
a\langle \exp c\rangle\langle b\rangle &= a\langle b\rangle + a\langle c\rangle\langle b\rangle + \frac{1}{2!}a\langle c,c\rangle\langle b\rangle + \frac{1}{3!}a\langle c,c,c\rangle\langle b\rangle + \cdots \\
&= a\langle b\rangle \\
&\quad + a\langle b, c\rangle + a\langle c\langle b\rangle\rangle \\
&\quad + \frac{1}{2!}\left(a\langle b,c,c\rangle + 2a\langle c\langle b\rangle, c\rangle\right) \\
&\quad + \frac{1}{3!}\left(a\langle b,c,c,c\rangle + 3a\langle c\langle b\rangle, c, c\rangle\right) + \cdots \\
&= a\langle b\rangle + a\langle b,c\rangle + \frac{1}{2!}a\langle b,c,c\rangle + \frac{1}{3!}a\langle b,c,c,c\rangle + \cdots \\
&\quad + a\langle c\langle b\rangle\rangle + a\langle c\langle b\rangle, c\rangle + \frac{1}{2!}a\langle c\langle b\rangle, c, c\rangle + \frac{1}{3!}a\langle c\langle b\rangle, c,c,c\rangle + \cdots \\
&= a\langle c\langle b\rangle, \exp c\rangle + a\langle b, \exp c\rangle.
\end{aligned}$$

This proves (19). Equation (20) can be proved similarly. □

Proof of Theorem 19. Let us apply the differential ∂ to the p-kernel $\boldsymbol{p}$ defined by (18). We obtain

$$\begin{aligned}
\partial \boldsymbol{p} = -\bar{l}\langle \bar{l}\rangle\langle \exp(h\circ \boldsymbol{p})\rangle + \bar{l}\langle \boldsymbol{p}, \exp(h\circ\boldsymbol{p})\rangle \\
-\bar{l}\langle gf\circ\boldsymbol{p}, \exp(h\circ\boldsymbol{p})\rangle - \bar{l}\langle h\circ\boldsymbol{p}\langle gf\circ\boldsymbol{p}\rangle, \exp(h\circ\boldsymbol{p})\rangle
\end{aligned} \tag{21}$$

In the above equation we used $\partial \bar{l} = \bar{l}\langle \bar{l} \rangle$ and the high school formula

$$\partial(\exp(h \circ \boldsymbol{p})) = \partial(h \circ \boldsymbol{p}) \exp(h \circ \boldsymbol{p}),$$

where, of course,

$$\partial(h \circ \boldsymbol{p}) = \partial h \circ \boldsymbol{p} + h \circ \partial \boldsymbol{p},$$

with $\partial h = \mathbb{1} - gf$ and $\partial \boldsymbol{p} = -\boldsymbol{p}\langle gf \circ \boldsymbol{p} \rangle$ by induction.

By the definition (18) of $\boldsymbol{p}$, the second term in the right hand side of (21) equals

$$\bar{l}\langle \bar{l}\langle \exp(h \circ \boldsymbol{p})\rangle, \exp(h \circ \boldsymbol{p})\rangle,$$

while equation (20) of Lemma 20, with $a = b = \bar{l}$ and $c = h \circ \boldsymbol{p}$, gives

$$-\bar{l}\langle \bar{l} \rangle \langle \exp(h \circ p)\rangle + \bar{l}\langle \bar{l}\langle \exp(h \circ \boldsymbol{p})\rangle, \exp(h \circ \boldsymbol{p})\rangle = 0.$$

Therefore the first two terms in the right hand side of (21) cancel. The remaining two terms combine, by equation (19) of Lemma 20 with $a = \bar{l}$, $b = gf \circ \boldsymbol{p}$ and $c = h \circ \boldsymbol{p}$, into $-\bar{l}\langle \exp(h \circ p)\rangle \langle gf \circ \boldsymbol{p}\rangle$. Since $\bar{l}\langle \exp(h \circ \boldsymbol{p})\rangle = \boldsymbol{p}$ by (18), equation (21) implies that $\partial \boldsymbol{p} = -\boldsymbol{p}\langle gf \circ \boldsymbol{p}\rangle$, which is (16). To prove that $\overline{k}$ defined by the Anzatz (17) fulfills $\overline{k}\langle \overline{k} \rangle = 0$ is equally simple as in the A_∞-case discussed in the first part of this section. □

Although we know from Proposition 17 that any symmetric brace algebra is generated by its pre-Lie part, it is unclear how to write the defining formula (15) for $\boldsymbol{p}$ using the pre-Lie multiplication only. We believe that this demonstrates that symmetric brace algebras are a useful concept even if they are "formally" the same as pre-Lie algebras. Since the operadic cochain complex of an arbitrary algebra over a quadratic Koszul operad $\mathcal{P}$ carries a natural symmetric brace algebra structure, the formulas of Theorem 19 in fact define transfers for *arbitrary* strongly homotopy $\mathcal{P}$-algebras.

4. The L_∞-structure of a Gauge Algebra

In this section we provide an example of how symmetric brace algebras may be used to organize the consequences of the algebraic assumptions that lead to a particular L_∞-algebra structure. This algebraic data, originally described by Berends, Burgers, and van Dam in their analysis of a particular type of gauge theory [1], was recast in [4] and was shown to lead to an L_∞-algebra structure on the space of gauge parameters together with the space of fields.

We summarize this situation in the following fashion. Let Ξ denote the (nongraded) vector space of gauge parameters and Φ denote the (nongraded) vector space of fields. The "action" is given by a gauge transformation which is phrased as a linear map $\delta : \Xi \to Hom(S^*(\Phi), \Phi)$ where $S^*(\Phi)$ is the cofree nilpotent cocommutative coassociative coalgebra cogenerated by Φ, which in fact coincides with the linear dual of the algebra of polynomial functions on Φ, see [14, Example II.3.79]. Let us remark that $S^*(\Phi)$ was, in [4], denoted ${}^c\!\bigwedge^*\Phi$. This notation was formally correct, but we think it could be easily mistaken for the exterior (Grassmann) coalgebra cogenerated by Φ. The map δ is then extended to a map $\hat{\delta} : Hom(S^*(\Phi), \Xi) \to Hom(S^*(\Phi), \Phi)$ via

$$\hat{\delta}(\pi) := ev \circ [(\delta \circ \pi) \otimes \mathbb{1}] \circ \Delta$$

where $ev : Hom(S^*(\Phi), \Phi) \otimes S^*(\Phi) \to \Phi$ is the evaluation map and Δ is the comultiplication on $S^*(\Phi)$. Recall that the vector space $Hom(S^*(\Phi), \Phi)$ has a canonical Lie bracket given by $[f, g] = f \circ \bar{g} - g \circ \bar{f}$ where $\bar{f}$ denotes the extension of a linear map $f \in Hom(S^*(\Phi), \Phi)$ to a coderivation $\bar{f}$ on $S^*(\Phi)$.

Another ingredient in this picture is the assumed existence of a map $C : \Xi \otimes \Xi \to Hom(S^*(\Phi), \Xi)$ that satisfies the (BBvD) hypothesis

$$[\delta(\xi), \delta(\eta)] = \hat{\delta}C(\xi, \eta) \in Hom(S^*(\Phi), \Phi)$$

for all $\xi, \eta \in \Xi$. After this map is extended to a map $\hat{C} : Hom(S^*(\Phi), \Xi) \otimes Hom(S^*(\Phi), \Xi) \to Hom(S^*(\Phi), \Xi)$,

$$\hat{C}(\pi_1, \pi_2) = ev \circ [(C \otimes \mathbb{1}) \circ (\pi_1 \otimes \pi_2 \otimes \mathbb{1})] \circ (\Delta \otimes \mathbb{1}) \circ \Delta, \tag{22}$$

a bracket that satisfies the Jacobi identity may be imposed on the space $Hom(S^*(\Phi), \Xi)$ via

$$[\pi_1, \pi_2] = \pi_1 \circ \overline{\hat{\delta}(\pi_2)} - \pi_2 \circ \overline{\hat{\delta}(\pi_1)} + \hat{C}(\pi_1, \pi_2).$$

Next, we consider the graded vector space $\mathbb{L}$ with $\mathbb{L}_0 := \Xi$, $\mathbb{L}_{-1} := \Phi$, and $\mathbb{L}_n := 0$ for $n \neq 0, -1$. By Theorem 2 of [4], an L_∞-structure may be defined on $\mathbb{L}$ by constructing a degree -1 map $D : {}^c\!\bigwedge^*(\uparrow \mathbb{L}) \to \uparrow \mathbb{L}$ by piecing together the maps δ and C. The Jacobi identity for the bracket on $Hom(S^*(\Phi), \Xi)$ implies that $D \circ \overline{D} = 0$, the L_∞ relations for $\mathbb{L}$.

In the remainder of this note, we show how this L_∞-structure may be explained in terms of the symmetric brace algebra $B_*(\mathbb{L})$ on $Hom(\mathbb{L}^{\otimes *}, \mathbb{L})^{as}$ constructed in Example 5. Specifically, we will use the symmetric brace algebra structure to construct a bracket on $Hom(\mathbb{L}^{\otimes *}, \mathbb{L})^{as}$ so that the existence of an L_∞-structure on $\mathbb{L}$ will be equivalent to this bracket satisfying the Jacobi identity on the subspace $Hom(\Phi^{\otimes *}, \Xi)^{as}$.

Let us fix two maps, ∇ and Υ in $Hom(\mathbb{L}^{\otimes *}, \mathbb{L})^{as}$. We require that

(i) the map ∇ have values in Φ and be the zero map when the number of inputs from the space Ξ is not equal to 1. Similarly, we require that
(ii) the map Υ take values in Ξ and be the zero map when the number of inputs from Ξ is not equal to 2.

An important example of maps with the above properties is provided by the maps δ and C, via the exponential correspondence

$$Hom(\Xi, Hom(S^*(\Phi), \Phi)) \ni \delta \longleftrightarrow \nabla \in Hom(\Xi \otimes S^*(\Phi), \Phi)$$

and

$$Hom(\Xi \wedge \Xi, Hom(S^*(\Phi), \Xi)) \ni C \longleftrightarrow \Upsilon \in Hom(\Xi \wedge \Xi \otimes S^*(\Phi), \Xi),$$

where $\wedge$ denotes the antisymmetric (exterior) product.

It follows immediately from (i) and (ii) above that $\nabla, \Upsilon \in B_{-1}(\mathbb{L})$. For $\alpha, \beta \in B_*(\mathbb{L})$, we define a degree -1 bracket

$$[\alpha, \beta] = \alpha\langle\nabla\langle\beta\rangle\rangle + (-1)^{\alpha\beta}\beta\langle\nabla\langle\alpha\rangle\rangle + \Upsilon\langle\alpha, \beta\rangle.$$

A pictorial definition of this bracket is given in Figure 1. We have

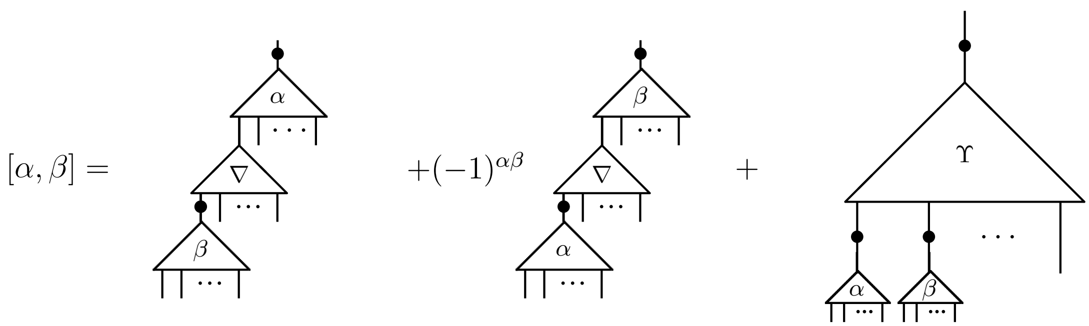


FIGURE 1. A pictorial description of $[\alpha, \beta]$. Multilinear maps are symbolized by triangles, with the bullet $\bullet$ marking outputs/inputs in Ξ.

Theorem 21. *The above bracket, restricted to* $Hom(\Phi^{\otimes *}, \Xi)^{\rm as} \subset B_1(V)$*, satisfies the Jacobi identity if and only if*

$$\nabla\langle\nabla\rangle + \nabla\langle\Upsilon\rangle = 0 \tag{23}$$

and

$$\Upsilon\langle\nabla\rangle + \Upsilon\langle\Upsilon\rangle = 0. \tag{24}$$

Proof. When we iterate the bracket, we obtain the expression

$$\begin{aligned}[\alpha, [\beta, \gamma]] \;\; = \;\; & \alpha\langle\nabla\langle\beta\langle\nabla\langle\gamma\rangle\rangle\rangle\rangle - \alpha\langle\nabla\langle\gamma\langle\nabla\langle\beta\rangle\rangle\rangle\rangle + \alpha\langle\nabla\langle\Upsilon\langle\beta, \gamma\rangle\rangle\rangle \\ & -\beta\langle\nabla\langle\gamma\rangle\rangle\langle\nabla\langle\alpha\rangle\rangle + \gamma\langle\nabla\langle\beta\rangle\rangle\langle\nabla\langle\alpha\rangle\rangle - \Upsilon\langle\beta, \gamma\rangle\langle\nabla\langle\alpha\rangle\rangle \\ & +\Upsilon\langle\alpha, \beta\langle\nabla\langle\gamma\rangle\rangle\rangle - \Upsilon\langle\alpha, \gamma\langle\nabla\langle\beta\rangle\rangle\rangle + \Upsilon\langle\alpha, \Upsilon\langle\beta, \gamma\rangle\rangle\end{aligned}$$

plus the corresponding terms with α, β, γ cyclicly permuted. There are two situations to consider.

We first examine the terms that have α outside of the braces. Two such terms that contain two ∇'s are exhibited explicitly above while two more may be found in the cyclic permutations of the above expression. We let

$$A = \alpha\langle\nabla\langle\beta\langle\nabla\langle\gamma\rangle\rangle\rangle\rangle - \alpha\langle\nabla\langle\gamma\langle\nabla\langle\beta\rangle\rangle\rangle\rangle - \alpha\langle\nabla\langle\beta\rangle\rangle\langle\nabla\langle\gamma\rangle\rangle + \alpha\langle\nabla\langle\gamma\rangle\rangle\langle\nabla\langle\beta\rangle\rangle$$

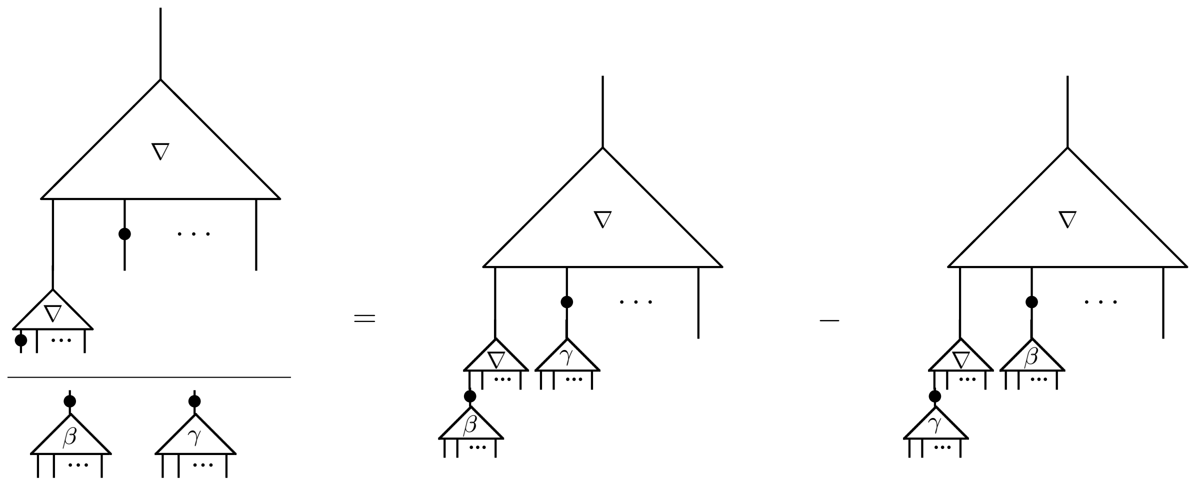


FIGURE 2. Equation $\nabla\langle\nabla\rangle\langle\beta,\gamma\rangle = \nabla\langle\nabla\langle\beta\rangle,\gamma\rangle - \nabla\langle\nabla\langle\gamma\rangle,\beta\rangle$. The horizontal line in the left hand side indicates that $\nabla\langle\nabla\rangle$ is performed first.

We apply the second relation in the definition of symmetric brace algebra twice to the third and fourth terms of A to obtain

$$-\alpha\langle\nabla\langle\beta\rangle\rangle\langle\nabla\langle\gamma\rangle\rangle = -\alpha\langle\nabla\langle\beta\rangle,\nabla\langle\gamma\rangle\rangle - \alpha\langle\nabla\langle\beta,\nabla\langle\gamma\rangle\rangle\rangle - \alpha\langle\nabla\langle\beta\langle\nabla\langle\gamma\rangle\rangle\rangle\rangle$$

and

$$\alpha\langle\nabla\langle\gamma\rangle\rangle\langle\nabla\langle\beta\rangle\rangle = \alpha\langle\nabla\langle\gamma\rangle,\nabla\langle\beta\rangle\rangle + \alpha\langle\nabla\langle\gamma,\nabla\langle\beta\rangle\rangle\rangle + \alpha\langle\nabla\langle\gamma\langle\nabla\langle\beta\rangle\rangle\rangle\rangle.$$

After adding these to the first two terms of A, we have

$$A = \alpha\langle\nabla\langle\nabla\langle\beta\rangle,\gamma\rangle\rangle - \alpha\langle\nabla\langle\nabla\langle\gamma\rangle,\beta\rangle\rangle.$$

Consider the symmetric algebra relation

$$\nabla\langle\nabla\rangle\langle\beta,\gamma\rangle = \nabla\langle\nabla\langle\beta,\gamma\rangle\rangle + \nabla\langle\nabla\langle\beta\rangle,\gamma\rangle - \nabla\langle\nabla\langle\gamma\rangle,\beta\rangle + \nabla\langle\nabla,\beta,\gamma\rangle$$

in which the first and the last summands are equal to zero because in each, the map ∇ has more than one input from Ξ. Consequently,

$$\nabla\langle\nabla\rangle\langle\beta,\gamma\rangle = \nabla\langle\nabla\langle\beta\rangle,\gamma\rangle - \nabla\langle\nabla\langle\gamma\rangle,\beta\rangle, \tag{25}$$

therefore

$$A = \alpha\langle\nabla\langle\nabla\rangle\langle\beta,\gamma\rangle\rangle.$$

Equation (25) is illustrated in Figure 2. The remaining term with α outside of the braces, $\alpha\langle\nabla\langle\Upsilon\langle\beta,\gamma\rangle\rangle\rangle$, may be replaced by $\alpha\langle\nabla\langle\Upsilon\rangle\langle\beta,\gamma\rangle\rangle$ because by using the symmetric brace relation, we have

$$\alpha\langle\nabla\langle\Upsilon\rangle\langle\beta,\gamma\rangle\rangle = \alpha\langle\nabla\langle\Upsilon\langle\beta,\gamma\rangle\rangle\rangle + \alpha\langle\nabla\langle\Upsilon\langle\beta\rangle,\gamma\rangle\rangle - \alpha\langle\nabla\langle\Upsilon\langle\gamma\rangle,\beta\rangle\rangle + \alpha\langle\nabla\langle\Upsilon,\beta,\gamma\rangle\rangle$$

in which the last three terms are equal to zero, because in each, the map ∇ has more than one input from Ξ. When we add this term to A, we have

$$\alpha\langle\nabla\langle\nabla\rangle\langle\beta,\gamma\rangle\rangle + \alpha\langle\nabla\langle\Upsilon\rangle\langle\beta,\gamma\rangle\rangle \tag{26}$$

together with the similar terms arising from the cyclic permutations of α, β, and γ.

To account for the remaining terms, we use arguments similar to those above to rewrite them as

$$-\Upsilon\langle\nabla\rangle\langle\alpha,\beta,\gamma\rangle - \Upsilon\langle\Upsilon\rangle\langle\alpha,\beta,\gamma\rangle. \tag{27}$$

Finally, we see that the Jacobi expression is equal to zero if both (26) and (27) are equal to zero, for all α, β and γ. It is not difficult to see that this happens if and only if (23) and (24) are satisfied (there are "enough test functions"). □

We note that equation (23) now plays the role of the (BBvD) hypothesis mentioned above. The following corollary is basically Theorem 2 of [4].

Corollary 22. *If the bracket (22) satisfies the Jacobi identity, then the map $l = \nabla + \Upsilon$ is an L_∞-structure for $\mathbb{L}$.*

Proof. As we know from Exercise 6, we must prove that $l\langle l\rangle = 0$. But this easily follows from

$$l\langle l\rangle = (\nabla+\Upsilon)\langle\nabla+\Upsilon\rangle = (\nabla\langle\nabla\rangle + \nabla\langle\Upsilon\rangle) + (\Upsilon\langle\nabla\rangle + \Upsilon\langle\Upsilon\rangle)$$

and equations (23) and (24). □

References

[1] F.A. Berends, G.J.H. Burgers, and H. van Dam, *On the theoretical problems in constructing intereactions involving higher spin massless particles*, Nucl.Phys.B **260** (1985), 295–322.

[2] F. Chapoton and M. Livernet. Pre-Lie algebras and the rooted trees operad. *Internat. Math. Res. Notices*, 8:395–408, 2001.

[3] M. Daily, T. Lada, *Symmetrization of brace algebras*, Preprint`math.QA/0503464`.

[4] R. Fulp, T. Lada, and J. Stasheff, *Sh-Lie algebras Induced by Gauge Transformations*, Comm in Math Phys. 231 (2002), 25–43.

[5] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. 78(2) (1963), 267–288.

[6] M. Gerstenhaber and A.A. Voronov, *Homotopy G-algebras and Moduli Space Operad*, Intern. Math. Res. Notices 3 (1995), 141–153.

[7] E. Getzler, *Cartan homotopy formulas and the Gauss-Manin connection in cyclic homology*, Israel Math. Conf. Proc. 7 (1993), 65-78.

[8] D. Guin and J.-M. Oudom, *On the Lie enveloping algebra of a pre-Lie algebra*. Preprint `math.QA/0404457`.

[9] T. Kadeishvili, *The structure of the $A(\infty)$-algebra and the Hochschild and Harrison cohomologies*, Trudi Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR 91 (1988), 19-27.

[10] T. Kimura, A. Voronov, and G. Zuckerman. *Homotopy Gerstenhaber algebra and topological field theory*, Operads: Proceedings of Renaissance Conference, Contemp. Math. 202 (1997), 305-333.

[11] T. Lada and M. Markl, *Strongly homotopy Lie algebras*, Comm. in Algebra (1995), 2147–2161.

[12] M. Markl. Homotopy algebras are homotopy algebras. *Forum Mathematicum*, 16(1):129–160, January 2004. Preprint available as `math.AT/9907138`.

[13] M. Markl. Transferring A_∞ (strongly homotopy associative) structures. Preprint `math.AT/0401007`, January 2004.

[14] M. Markl, S. Shnider, and J. D. Stasheff. *Operads in Algebra, Topology and Physics*, volume 96 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, Rhode Island, 2002.

[15] M. Schlessinger and J.D. Stasheff, *The Lie algebra structure of tangent cohomology and deformation theory*, J. Pure Appl. Algebra 38 (1985), 313–322.

Department of Mathematics, North Carolina State University, Raleigh NC 27695, USA

E-mail address: `lada@math.ncsu.edu`

Mathematics Institute of the Academy, Žitná 25, 115 67 Praha 1, The Czech Republic

E-mail address: `markl@math.cas.cz`